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Post-Einstein Deflection of Light in the Vicinity of a Spherical Star

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Abstract

An extended Riemannian gravitational scalar potential exterior to the body for static homogeneous spherical massive bodies is used to obtain an extended Riemannian world-line element. The extended Riemannian world-line element was solved to derive the photon equations of motion. The photon equations of motion were solved by seeking the complementary and particular solutions to obtain a deflection equation for starlight grazing the edge of the sun. The obtained generalized deflection equations reduce in the order of c^0 *to the corresponding pure Einstein's equation and to the order of* c^{-2} *it contains post Einstein terms. The calculated value of the deflection angle for the Sun is found to be* $1.736''$ *which is in good agreement with the experimentally accepted Einstein's value of* $1.75''$. *This result point to the fact that the extended Riemannian gravitational scalar potential exterior to the body introduced in this research work can effectively be used to obtain the deflection equation as excellently as Einstein's geometrical theory of gravitation.*

Keywords: Riemannian, gravitational, scalar potential, Photon, Spherical Body, Deflection

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INTRODUCTION

Studies on the deflection of light by massive bodies can be traced to the 18th century. Newton gave a pioneer mathematical description of gravity and suggested that light particles are affected by gravity in the same way as ordinary matter. The German astronomer Johann Georg von Soldner published the first calculation of the deflection of light by mass. He showed that rays from a distant star skimming the Sun's surface would be deflected through an angle of about 0.9 seconds of arc, or one quarter of a thousandth of a degree. Soldner's calculations were based on Newton's laws of motion and gravitation, and the assumption that light behaves like very fast moving particles (Steven & Irwin, 2010; Anderson, 1967; Weinberg, 1972).

In the 20th century, Einstein developed the theory of general relativity. He calculated that the deflection of light predicted by his theory would be twice the Newtonian value. According to the general theory of relativity, a ray of light arriving from the left would be bent inwards such that its apparent direction of origin, when viewed from the right, would differ by an angle (the deflection angle) whose magnitude is inversely proportional to the distance of the closest approach of the ray path to the center of mass (Anderson, 1967; Weinberg, 1972; Howusu, 2010; Ndikilar, & Howusu, 2009).

The first successful measurement of gravitational deflection of light was in May 1919, by two British expeditions organized and sponsored by the Royal Astronomical Society and the Royal Society. The results of the measurements showed that light was deflected, and that this deflection was observed to be $(1.60 \pm 0.31)''$ which was more consistent with general relativity than Newtonian theory. In may, 1926 at Sumatra, June, 1936 in USSR and May, 1947 at Brazil in Yerkes observatory a deflection values of $(2.24 \pm 0.10)''$, $(2.73 \pm 0.31)''$ and $(2.01 \pm 0.27)''$ was observed (Anderson, 1967; Weinberg, 1972).

In 2009, an entirely new approach to the search for the metric tensor of space time with the influence and interaction of gravitational fields was introduced (Steven & Irwin, 2010; Howusu, 2010; Howusu, 2013; Lumbi *et al.,* 2014). In this approach, the metric tensor is a fundamental quantity of nature and can be obtained through extensions of the Euclidean metric tensor. In this paper, we introduce a new approach to derive the deflection equation in the gravitational field of a spherical star using an extended Riemannian metric tensor.

METHODOLOGY

The well known gravitational scalar potential of the Sun exterior (assumed to be a homogeneous spherical body) which Newton's and Einstein's equation of motion for photon were derived is given explicitly as [1 - 3]

$$f = \frac{k}{r} \tag{1}$$

The extended Riemannian gravitational scalar potential exterior to a spherical body (f) is shown to be given explicitly as (Howusu , 2010; Lumbi *et al.,* 2014)

$$f = \frac{k}{r}\left\{1 - \frac{k}{c^2 R}\right\} - \frac{k^2}{c^2 r^2} + \cdots \tag{2}$$

where, $k = GM$, G is the universal gravitational constant, M is the mass of the planets, c is the speed of light R is the radius of the spherical body and $r > R$ for the exterior field.
It must be noted that equation (1) was derived based on the well - known Euclidean geometry while equation (2) was derived based on the Riemannian geometry.

To obtain the generalized general relativistic equation of photon, the extended Riemannian gravitational scalar potential exterior (assumed to be a homogeneous spherical body) is substituted into the world line element to obtain an extended Riemannian world-line element. The extended Riemannian world-line element was solved to derive a generalized photon equation of motion. The photon equation of motion was solved by seeking the complementary and particular solution to obtain deflection equation for starlight grazing the edge of the sun (assumed to spherical in shape).

RESULTS AND DISCUSSION

Consider the motion of a particle whose motion is confine to the equatorial plane of the spherical body (e.g Sun, planet, comet or asteroid), then from geometry considerations in spherical polar coordinates

$$\theta = \frac{\pi}{2}$$

According to the General Relativity Theory, light (photon) moves along a null geodesic. Therefore,

$$c^2 d\tau^2 = 0 \tag{3}$$

The general expression for the line element in the gravitational field of spherical massive body is given explicitly from the extended metric tensor by (Anderson, 1967; Weinberg, 1972; Lumbi *et al.*, 2020)

$$c^2 d\tau^2 = -c^2 \left\{1 - \frac{2}{c^2} f\left(r, \theta, \phi, x^{/0}\right)\right\} dt^2 - \left\{1 - \frac{2}{c^2} f(r, \theta, \phi, x^0)\right\}^{-1} dr^2 - r^2 d\phi^2 - r^2 sin^2\theta d\phi^2 \tag{4}$$

Substituting equations (1) and (2) into (3) we obtain

$$c^2 \left\{1 - \frac{2}{c^2}\left[\frac{GM}{r}\left(1 - \frac{GM}{c^2 R}\right) - \frac{G^2 M^2}{c^2 r^2}\right]\right\} dt^2 - \left\{1 - \frac{2}{c^2}\left[\frac{GM}{r}\left(1 - \frac{GM}{c^2 R}\right) - \frac{G^2 M^2}{c^2 r^2}\right]\right\}^{-1} dr^2 - r^2 d\phi^2 = 0 \tag{5}$$

where, M is the mass of the body, G is the universal gravitational constant, c is the speed of light in vaccum, r is the distance from the centre of the body and R is the radius of the body.

Equation (5) is the extended Riemannian metric world-line element. The extended Riemannian world-line element contains additional correctional terms which are not found in the existing world-line element. This is as the result of the additional correctional terms in the extended Riemannian gravitational scalar potential exterior to spherical bodies.

Let ω be a parameter which may be used to follow the motion of a photon. Dividing through by $d\omega^2$ and let dot $(.)$ denote differentiation with respect to ω we get

$$\left\{1-\frac{2}{c^2}\left[\frac{GM}{r}\left(1-\frac{GM}{c^2R}\right)-\frac{G^2M^2}{c^2r^2}\right]\right\}d\dot{t}^2$$

$$c^2-\left\{1-\frac{2}{c^2}\left[\frac{GM}{r}\left(1-\frac{GM}{c^2R}\right)-\frac{G^2M^2}{c^2r^2}\right]\right\}^{-1}d\dot{r}^2-r^2d\dot{\phi}^2=0 \quad (6)$$

Considering a clock at rest in the gravitational field, it can be shown that that

$$\dot{t}=a\left\{1-\frac{2}{c^2}\left[\frac{GM}{r}\left(1-\frac{GM}{c^2R}\right)-\frac{G^2M^2}{c^2r^2}\right]\right\}^{-1} \quad (7)$$

Also, the pure azimuthal speed can be deduced as

$$\dot{\phi}=\frac{b}{r^2} \quad (8)$$

where, l is a constant of motion, which physically corresponds to the angular momentum per unit mass. Also, $\dot{r}$ can be written in terms ϕ as follows

$$\dot{r}=\frac{dr}{d\phi}.\frac{d\phi}{d\omega}=\dot{\phi}\frac{dr}{d\phi}=\frac{b}{r^2}\frac{dr}{d\phi}$$

Hence,

$$(\dot{r})^2=\frac{b^4}{r^4}\left(\frac{dr}{d\phi}\right)^2 \quad (9)$$

Substituting equations (5) - (7) into (4) yields

$$\frac{b^2}{r^4}\left(\frac{dr}{d\phi}\right)^2=-\frac{b^2}{r^2}\left\{1-\frac{2}{c^2}\left[\frac{GM}{r}\left(1-\frac{GM}{c^2R}\right)-\frac{G^2M^2}{c^2r^2}\right]\right\}+a^2c^2 \quad (10)$$

But

$$r=\frac{1}{u}\Rightarrow\frac{dr}{d\phi}=-u^{-2}\frac{du}{d\phi}$$

$$\left(\frac{du}{d\phi}\right)^2=u^4\left(\frac{dr}{d\phi}\right)^2 \quad (11)$$

Using equation (11) in (10), we obtain

$$\left(\frac{dr}{d\phi}\right)^2=u^4\left[\frac{a^2c^2}{b^2}r^4-r^2\left\{1-\frac{2}{c^2}\left[\frac{GM}{r}\left(1-\frac{GM}{c^2R}\right)-\frac{G^2M^2}{c^2r^2}\right]\right\}\right]$$

By simplification, we obtain

$$\left(\frac{dr}{d\phi}\right)^2=\frac{a^2c^2}{b^2}-\frac{1}{r^2}\left\{1-\frac{2}{c^2}\left[\frac{GM}{r}\left(1-\frac{GM}{c^2R}\right)-\frac{G^2M^2}{c^2r^2}\right]\right\} \quad (12)$$

or more explicitly

$$\left(\frac{dr}{d\phi}\right)^2=\frac{a^2c^2}{b^2}-u^2\left\{1-\frac{2}{c^2}\left[GMu\left(1-\frac{GM}{c^2R}\right)-\frac{G^2M^2}{c^2}u^2\right]\right\} \quad (13)$$

By expansion of the right hand side of equation (13) yields

$$\left(\frac{dr}{d\phi}\right)^2=\frac{a^2c^2}{b^2}-u^2-\left\{\frac{2}{c^2}\left[GMu\left(1-\frac{GM}{c^2R}\right)-\frac{G^2M^2}{c^2}\right]u^2\right\} \quad (14)$$

Differentiating both sides of equation (14) with respect to ϕ and simplifying gives

$$\frac{d^2u}{d\phi^2}+u=\left\{\frac{3}{c^2}\left[GM\left(1-\frac{GM}{c^2R}\right)-\frac{G^2M^2}{c^2}\right]u^2\right\} \quad (15)$$

Equation (15) is the generalized general relativistic equation of a photon in the vicinity of a spherical body. This equation contains additional correctional terms not found in Newton's or Einstein's equations or calculations for static spherical fields (Rosser, 1964; Anderson, 1967; Weinberg, 1967; Ndikilar, 2009).
Equation (15) is a second order ordinary differential equation (ODE), which has the general solution of the form

$$u(\phi) = u_c(\phi) + u_p(\phi) \tag{16}$$

where, $u_c(\phi)$ and $u_p(\phi)$ is the complementary and particular solution respectively.
Solving the complementary solution by equating the right hand side to zero, we obtain

$$u_c(\phi) = u_0 \cos\phi \text{ or } u_c(\phi) = u_0 \sin\phi$$

Hence we consider the first solution given explicitly as

$$u_c(\phi) = u_0 \cos\phi \tag{17}$$

Equation (17) is the called complementary solution.
By the assumption that

$$u_p(\phi) = A\cos\phi + \mathrm{B}\sin\phi \tag{18}$$

where, A and B are arbitrary constants.
Differentiating $u_p(\phi)$ with respect to ϕ twice and substituting into equation and solving for $u_p(\phi)$, we obtain

$$u_p(\phi) = -\left[\frac{GM}{c^2}\left(1 - \frac{GM}{c^2R}\right) - \frac{G^2M^2}{c^2}\right] u_0(2 - \cos^2\phi) \tag{19}$$

Adding equations (17) and (19) we obtain the general solution given explicitly as

$$u = u_0\cos\phi - \left[\frac{GM}{c^2}\left(1 - \frac{GM}{c^2R}\right) - \frac{G^2M^2}{c^2}\right] u_0(2 - \cos^2\phi) \tag{20}$$

or

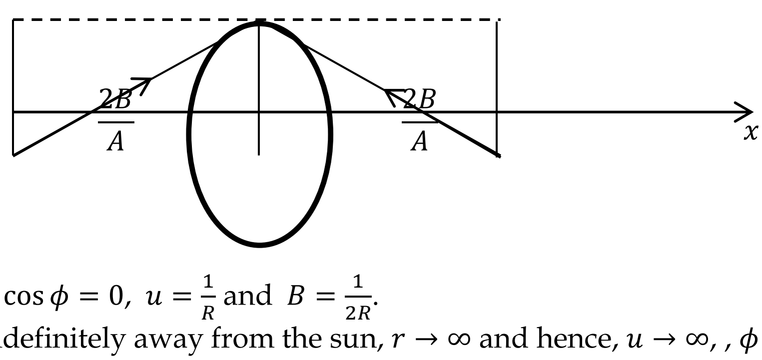
$$u = Ax + B_0(2 - \cos^2\phi) \tag{21}$$

where,

$$A = u_0,\ B = \left[\frac{GM}{c^2}\left(1 - \frac{GM}{c^2R}\right) - \frac{G^2M^2}{c^2}\right] u_0 \text{ and } x = \cos\phi$$

Equation (21) is the Cartesian equation for the conic section.
Consider a photon grazing the edge of the sun as seen in the figure 1 below

Fig. 1



At point Q, $\phi = \frac{\pi}{2},\ x = \cos\phi = 0,\ u = \frac{1}{R}$ and $B = \frac{1}{2R}$.
As the photon moves indefinitely away from the sun, $r \to \infty$ and hence, $u \to \infty$, , $\phi \to \phi_\infty$ and $x \to x_\infty$.

Then, equation (21) reduces to

$$0 = Ax_\infty + B_0(2 - x_\infty^2) \tag{22}$$

or

$$Bx_\infty^2 - Ax_\infty - 2B_0 = 0 \tag{23}$$

Equation (23) is a quadratic equation in x_∞^2. Solving equation (23) using quadratic formula, yields

$$x_\infty = \frac{A}{B} \tag{24}$$

or

$$x_\infty = -\frac{2B}{A} \tag{25}$$

Equation (24) is mathematically sound but physically of no significance as it yields a complex solution. Hence, we consider equation (25) as our physical expression.
From the second solution

$$cos\ \phi_\infty = -\frac{2B}{A}$$

or

$$cos\ \phi_\infty = \frac{\pi}{2} + \Delta_1 \tag{26}$$

where,

$$\Delta_1 = \frac{2B}{A}$$

Let Δ be the total deflection of the photon from the origin of the straight line motion. Then the total deflection of the photon from the origin of the straight line motion is given by

$$\Delta = \frac{4B}{A} = 4\left[\frac{GM}{c^2}\left(1 - \frac{GM}{c^2R}\right) - \frac{G^2M^2}{c^2}\right] \tag{27}$$

By expansion we obtain

$$\Delta = \frac{4GM}{c^2R} - \frac{4G^2M^2}{c^4R^2} - \frac{4G^2M^2}{c^2R} \tag{28}$$

Neglecting powers of c^{-4} and above, we obtain

$$\Delta = \frac{4GM}{c^2R} - \frac{4G^2M^2}{c^2R} \tag{29}$$

Equation (29) is the total deflection equation of a photon from its original straight line motion. To first order approximation equation (29) reduces to

$$\Delta = \frac{4GM}{c^2R} \tag{30}$$

Putting the value of universal gravitational constant, G = 6.672 * $10^{11}NM^{2/}Kg^2$, mass of the Sun, M_{Sun} = 1.989 * $10^{30}kg$, radius of the Sun, R_{Sun}= 6.690 * 10^8 and the speed of light c = 2.99979 ms^{-1} into equation (30) gives deflection angle of 1.736″**.** This result is in good agreement with Einstein's experimentally established value of 1.75″.

CONCLUSION

We have shown how to derive the extended Riemannian world-line element, general relativistic equation of motion and deflection equation of a photon using the extended Riemannian gravitational scalar potential exterior to the body for static homogeneous spherical massive bodies. The extended Riemannian world-line element, general relativistic equation and deflection equation of a photon are found to be equations (5), (15) and (30). It is interesting and instructive to note that these equations reduces to the corresponding pure Einstein in the order of c^0 and to the order of c^{-2} it contains post Einstein correction terms. The calculated value of the deflection angle in the vicinity of spherical star is found to be $1.736''$ which is within the experimentally accepted values of $(1.60 \pm 0.31)''$, $(2.24 \pm 0.10)''$, $(2.73 \pm 0.31)''$ and $(2.01 \pm 0.27)''$ observed in November, 1919 in London, May, 1926 at Sumatra, June, 1936 in USSR and May, 1947 at Brazil in Yerkes observatory.

The result obtained in this paper point to the fact that the Riemannian gravitational scalar potential exterior to the body introduced in this research work can effectively be used to obtain the deflection equation as excellently as Einstein's geometrical theory of gravitation. The door is therefore open up for theoretical development and experimental investigations and possible applications by Physicists.

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