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Solution of Einstein's G_{22} Field Equation Exterior to a Spherical Mass with Varying Potential

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Abstract
In general theory of relativity, Einstein's field equations relate the geometry of space-time with the distribution of matter within it. These equations were first published by Einstein in the form of a tensor equation which related the local space-time curvature with the local energy and momentum within this space-time. In this article, Einstein's geometrical field equations exterior to astrophysically real or hypothetical distribution of mass within a spherical geometry is constructed and solved for field whose gravitational potential varies with time, radial distance and polar angle. Sixteen affine connections from this field were used to construct Riemann-Christoffel tensors, the Ricci tensor and the Einstein's tensors and the solution of Einstein's field equation was obtained using power series. The metric tensors and the solution of the Einstein's field equations used in this work has only one arbitrary function $f(t,r,\theta)$*, and thus put the Einstein's geometrical theory of gravitation on the same bases with the Newton's dynamical theory of gravitation. The gravitational scalar potential* $f(t,r,\theta)$ *obtained in this research work to the order of* c^0*,* c^{-2} *contains Newton dynamical gravitational scalar potential and post Newtonian additional terms which can be applied to the study of rotating bodies such as stars.*

Keywords: Einstein's field equation, radial distance, polar angle, Schwarzchild's metric, tensor

INTRODUCTION
Gravitation is a natural phenomenon whose study gives a better understanding of the universe. On earth, it gives weight to physical objects and causes the ocean tides. The gravitational attraction of the original gaseous matter present in the Universe caused it to begin coalescing, forming stars and the stars to group together into galaxies. Gravity is responsible for many of the large-scale structures in the Universe (Green, 2004).

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After the publication of Einstein's geometrical gravitational field equations (EGGFE) in 1915, the search for their exact and analytical solutions for all the gravitational fields in nature began (Howusu, 2010; Chifu and Howusu, 2009; Chifu, 2012). Schwarzchild first constructed the exact solution to this field equation in static and pure radial spherical polar coordinates in 1916 by considering astrophysical bodies such as the sun and the stars (Lumbi, *et al.*, 2014) In Schwarzchild's metric, the tensor field varies with radial distance only.

A new method and approach was introduced to formulate exact analytical solutions (Chifu, 2009) as an extension of Schwarzschild's method. This new approach took into consideration the fact that tensor field of astrophysical bodies does not depend on radial distance only as indicated in Schwarzschild's equation. This new approach was used in several studies of Einstein's geometrical field equations (Chifu and Howusu, 2009; Chifu and Lumbi, 2008; Sarki, *et al.*,). This method would help in the study of Ceres, Pluto, Makemake, Haumea, The Ouort Cloud and other astrosphysical bodies. In this research work, we show how exact analytical solution of the exterior field equation can be constructed in the limit of c^{-2} in a gravitational field for time varying spherical massive bodies using the new method and approach.

CONSTRUCTION OF THE EXTERIOR G_{22} FIELD EQUATION

To construct the G_{22} field equation, we applied the covariant metric tensors for this distribution of mass or pressure in spherical polar coordinates $f(t,r,\theta)$ constructed by Lumbi, *et al.*,(2014); Howusu, (2007) and Howusu, (2009). It is a time varying metric tensor that depends on radial distance and polar angle and is given as

$$g_{00} = \left[1 + \frac{2f(t,r,\theta)}{c^2}\right] \quad (2.1)$$

$$g_{11} = -\left[1 + \frac{2f(t,r,\theta)}{c^2}\right]^{-1} \quad (2.2)$$

$$g_{22} = -r^2 \quad (2.3)$$

$$g_{33} = -r^2 \sin^2\theta \quad (2.4)$$

$$g_{\mu\nu} = 0, \text{ otherwise} \quad (2.5)$$

where

$f(t,r,\theta)$ is an arbitrary function, determined by the mass or pressure and possess symmetries of the latter. In approximate gravitational fields, it is equal to Newton's gravitational scalar potential exterior to the spherical mass distribution.

To obtain the corresponding contravariant metric tensors for this gravitational field, the Quotient Theorem (Lumbi, *et al.*, 2014) of tensor analysis was used to obtain the components of the contravariant tensor as

$$g^{00} = \left[1 + \frac{2f(t,r,\theta)}{c^2}\right]^{-1} \quad (2.6)$$

$$g^{11} = -\left[1 + \frac{2f(t,r,\theta)}{c^2}\right] \quad (2.7)$$

$$g^{22} = -\frac{1}{r^2} \tag{2.8}$$

$$g^{33} = -\frac{1}{r^2 \sin^2 \theta} \tag{2.9}$$

$g^{\mu\nu} = 0$, otherwise (2.10)

The coefficients of affine connections, defined by the metric tensors of space-time are determined (Lumbi, *et al.,* 2014; Howusu, 2009; Arfken, 1995; Bergmann, 1987) using equations (2.1)-(2.10) and equation (2.11),

$$\Gamma^{\mu}{}_{\alpha\beta} = \frac{1}{2} g^{\mu\xi} \left(g_{\alpha\xi,\beta} + g_{\beta\xi,\alpha} - g_{\alpha\beta,\xi} \right) \tag{2.11}$$

They are found to be given explicitly in terms of (ct, r, θ)within this regionas

$$\Gamma^{0}{}_{00} = \frac{1}{c^2}\left[1 + \frac{2f(t,r,\theta)}{c^2}\right]^{-1} \frac{\partial f(t,r,\theta)}{\partial t} \tag{2.12}$$

$$\Gamma^{0}{}_{01} = \Gamma^{0}{}_{10} = \frac{1}{c^2}\left[1 + \frac{2f(t,r,\theta)}{c^2}\right]^{-1} \frac{\partial f(t,r,\theta)}{\partial r} \tag{2.13}$$

$$\Gamma^{0}{}_{11} = -\frac{1}{c^2}\left[1 + \frac{2f(t,r,\theta)}{c^2}\right]^{-3} \frac{\partial f(t,r,\theta)}{\partial t} \tag{2.14}$$

$$\Gamma^{0}{}_{02} = \Gamma^{0}{}_{20} = \frac{1}{c^2}\left[1 + \frac{2f(t,r,\theta)}{c^2}\right]^{-1} \frac{\partial f(t,r,\theta)}{\partial \theta} \tag{2.15}$$

$$\Gamma^{1}{}_{00} = \frac{1}{c^2}\left[1 + \frac{2f(t,r,\theta)}{c^2}\right] \frac{\partial f(t,r,\theta)}{\partial r} \tag{2.16}$$

$$\Gamma^{1}{}_{01} = \Gamma^{1}{}_{10} = -\frac{1}{c^2}\left[1 + \frac{2f(t,r,\theta)}{c^2}\right]^{-1} \frac{\partial f(t,r,\theta)}{\partial t} \tag{2.17}$$

$$\Gamma^{1}{}_{11} = -\frac{1}{c^2}\left[1 + \frac{2f(t,r,\theta)}{c^2}\right]^{-1} \frac{\partial f(t,r,\theta)}{\partial r} \tag{2.18}$$

$$\Gamma^{1}{}_{12} = \Gamma^{1}{}_{21} = -\frac{1}{c^2}\left[1 + \frac{2f(t,r,\theta)}{c^2}\right]^{-1} \frac{\partial f(t,r,\theta)}{\partial \theta} \tag{2.19}$$

$$\Gamma^{1}{}_{22} = -r\left[1 + \frac{2f(t,r,\theta)}{c^2}\right] \tag{2.20}$$

$$\Gamma^{1}{}_{33} = -r\sin^2\theta\left[1 + \frac{2f(t,r,\theta)}{c^2}\right] \tag{2.21}$$

$$\Gamma^{2}{}_{00} = \frac{1}{c^2 r^2} \frac{\partial f(t,r,\theta)}{\partial \theta} \tag{2.22}$$

$$\Gamma^{2}{}_{11} = \frac{1}{c^2 r^2}\left[1 + \frac{2f(t,r,\theta)}{c^2}\right]^{-2} \frac{\partial f(t,r,\theta)}{\partial \theta} \tag{2.23}$$

$$\Gamma^2{}_{12} = \Gamma^2{}_{21} = \frac{1}{r} \tag{2.24}$$

$$\Gamma^2{}_{33} = -\sin\theta\cos\theta \tag{2.25}$$

$$\Gamma^3{}_{13} = \Gamma^3{}_{31} = \frac{1}{r} \tag{2.26}$$

$$\Gamma^3{}_{23} = \Gamma^3{}_{32} = \cot\theta \tag{2.27}$$

$$\Gamma^\mu{}_{\alpha\beta} = 0; \text{ otherwise} \tag{2.28}$$

The exterior G_{22} field equation in this field is given as

$$G_{22} = R_{22} - \frac{1}{2}Rg_{22} = 0 \tag{2.29}$$

The choice of this component is because it's observed that all the solution to the field equation towards the exterior converges at the same way.

The expression for the Ricci tensor R_{22} and the curvature scalar R in this field are given respectively as:

$$R_{22} = R^0_{220} + R^1_{221} + R^2_{222} + R^3_{223} \tag{2.30}$$

$$R = g^{00}R_{00} + g^{11}R_{11} + g^{22}R_{22} + g^{33}R_{33} \tag{2.31}$$

The expanded form of equations (2.30) and (2.31) are given as

$$\begin{aligned} R_{22} = {} & \Gamma^0{}_{20,2} + \Gamma^0{}_{20}\Gamma^0{}_{02} - \Gamma^1{}_{22}\Gamma^0{}_{10} + \Gamma^1{}_{21,2} - \Gamma^1{}_{22,1} + \Gamma^1{}_{21}\Gamma^1{}_{12} - \Gamma^1{}_{22}\Gamma^1{}_{11} - \Gamma^2{}_{12}\Gamma^1{}_{22} \\ & + \Gamma^2_{22,2} - \Gamma^2_{22,2} + \Gamma^0_{22}\Gamma^2_{02} - \Gamma^0_{22}\Gamma^2_{02} + \Gamma^1_{22}\Gamma^2_{12} - \Gamma^1_{22}\Gamma^2_{12} + \Gamma^2_{22}\Gamma^2_{22} - \Gamma^2_{22}\Gamma^2_{22} + \Gamma^3_{22}\Gamma^2_{32} - \Gamma^3_{22}\Gamma^2_{32} \\ & + \Gamma^3{}_{23,2} - \Gamma^1{}_{22}\Gamma^3{}_{13} + \Gamma^3{}_{23}\Gamma^3{}_{32} \end{aligned} \tag{2.32}$$

$$\begin{aligned} R = {} & g^{00}\{\Gamma^0_{00,0} - \Gamma^0_{00,0} + \Gamma^0_{00}\Gamma^0_{00} - \Gamma^0_{00}\Gamma^0_{00} + \Gamma^1_{00}\Gamma^0_{10}\Gamma^1{}_{01,0} - \Gamma^1_{00}\Gamma^0_{10} \\ & - \Gamma^2_{00}\Gamma^0{}_{20} - \Gamma^2_{00}\Gamma^0{}_{20} + \Gamma^3_{00}\Gamma^0{}_{30} - \Gamma^3_{00}\Gamma^0{}_{30} - \Gamma^1{}_{00,1} + \Gamma^0{}_{01}\Gamma^1{}_{00} - \Gamma^0{}_{00}\Gamma^1{}_{01} + \Gamma^1{}_{01}\Gamma^1{}_{10} \\ & - \Gamma^1{}_{00}\Gamma^1{}_{11} - \Gamma^2{}_{00}\Gamma^1{}_{21} - \Gamma^2{}_{00,2} + \Gamma^0{}_{02}\Gamma^2{}_{00} - \Gamma^1{}_{00}\Gamma^2{}_{12} - \Gamma^1{}_{00}\Gamma^3{}_{13} - \Gamma^2{}_{00}\Gamma^3{}_{23}\} \\ & + g^{11}\{\Gamma^0{}_{10,1} - \Gamma^0{}_{11,0} + \Gamma^0{}_{10}\Gamma^0{}_{01} - \Gamma^0{}_{11}\Gamma^0{}_{00} + \Gamma^1{}_{10}\Gamma^0{}_{11} - \Gamma^1{}_{11}\Gamma^0{}_{10} \\ & - \Gamma^2{}_{11}\Gamma^0{}_{20} + \Gamma^1_{11,1} - \Gamma^1_{11,1} + \Gamma^0_{11}\Gamma^1_{01} - \Gamma^0_{11}\Gamma^1_{01} + \Gamma^1_{11}\Gamma^1_{11} - \Gamma^1_{11}\Gamma^1_{11} + \Gamma^2_{11}\Gamma^1_{21} - \Gamma^2_{11}\Gamma^1_{21} \\ & + \Gamma^3_{11}\Gamma^1_{31} - \Gamma^3_{11}\Gamma^1_{31} + \Gamma^2{}_{12,1} - \Gamma^2{}_{11,2} + \Gamma^1{}_{12}\Gamma^2{}_{11} - \Gamma^1{}_{11}\Gamma^2{}_{12} + \Gamma^2{}_{12}\Gamma^2{}_{21} + \Gamma^3{}_{13,1} - \Gamma^1{}_{11}\Gamma^3{}_{13} - \Gamma^2{}_{11}\Gamma^3{}_{23} + \Gamma^3{}_{13}\Gamma^3{}_{31}\} \\ & + g^{22}\{\Gamma^0{}_{20,2} + \Gamma^0{}_{20}\Gamma^0{}_{02} - \Gamma^1{}_{22}\Gamma^0{}_{10} + \Gamma^1{}_{21,2} - \Gamma^1{}_{22,1} + \Gamma^1{}_{21}\Gamma^1{}_{12} - \Gamma^1{}_{22}\Gamma^1{}_{11} - \Gamma^2{}_{12}\Gamma^1{}_{22} \\ & + \Gamma^2_{22,2} - \Gamma^2_{22,2} + \Gamma^0_{22}\Gamma^2_{02} - \Gamma^0_{22}\Gamma^2_{02} + \Gamma^1_{22}\Gamma^2_{12} - \Gamma^1_{22}\Gamma^2_{12} + \Gamma^2_{22}\Gamma^2_{22} - \Gamma^2_{22}\Gamma^2_{22} + \Gamma^3_{22}\Gamma^2_{32} - \Gamma^3_{22}\Gamma^2_{32} \\ & + \Gamma^3{}_{23,2} - \Gamma^1{}_{22}\Gamma^3{}_{13} + \Gamma^3{}_{23}\Gamma^3{}_{32}\} + g^{33}\{-\Gamma^1{}_{00}\Gamma^3{}_{13} - \Gamma^2{}_{00}\Gamma^3{}_{23} - \Gamma^1{}_{33,1} - \Gamma^1{}_{33}\Gamma^1{}_{11} - \Gamma^2{}_{33}\Gamma^1{}_{21} \\ & + \Gamma^3{}_{31}\Gamma^1{}_{33} - \Gamma^2{}_{33,2} - \Gamma^1{}_{33}\Gamma^2{}_{12} + \Gamma^3{}_{32}\Gamma^2{}_{33} + \Gamma^3_{33,3} - \Gamma^3_{33,3} + \Gamma^0_{33}\Gamma^3_{03} - \Gamma^0_{33}\Gamma^3_{03} \\ & + \Gamma^1_{33}\Gamma^3_{13} - \Gamma^1_{33}\Gamma^3_{13} + \Gamma^2_{33}\Gamma^3_{23} - \Gamma^2_{33}\Gamma^3_{23} + \Gamma^3_{33}\Gamma^3_{33} - \Gamma^3_{33}\Gamma^3_{33}\} \end{aligned} \tag{2.33}$$

Explicitly equations (2.32) and (2.33) are given as

$$R_{22} = \frac{2}{c^4}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left(\frac{\partial f(t,r,\theta)}{\partial \theta}\right)^2 + \frac{2r}{c^2}\frac{\partial f(t,r,\theta)}{\partial r} + \frac{2f(t,r,\theta)}{c^2} \tag{2.34}$$

$$R = \frac{8}{c^4}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-3}\left[\frac{\partial f(t,r,\theta)}{\partial t}\right]^2 - \frac{8}{c^2 r}\frac{\partial f(t,r,\theta)}{\partial r} - \frac{2}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left(\frac{\partial^2 f(t,r,\theta)}{\partial t^2}\right)$$
$$-\frac{2}{c^2}\left(\frac{\partial^2 f(t,r,\theta)}{\partial r^2}\right) - \frac{2}{c^4 r^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left(\frac{\partial f(t,r,\theta)}{\partial \theta}\right)^2 - \frac{4f(t,r,\theta)}{c^2 r^2} \tag{2.35}$$

Substituting equations (2.3), (2.34) and (2.35) into (2.29) gives

$$G_{22} = \frac{2}{c^4}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left(\frac{\partial f(t,r,\theta)}{\partial \theta}\right)^2 + \frac{2r}{c^2}\frac{\partial f(t,r,\theta)}{\partial r} + \frac{2f(t,r,\theta)}{c^2}$$
$$+\frac{1}{2}r^2\left[\frac{8}{c^4}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-3}\left[\frac{\partial f(t,r,\theta)}{\partial t}\right]^2 - \frac{8}{c^2 r}\frac{\partial f(t,r,\theta)}{\partial r} - \frac{2}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left(\frac{\partial^2 f(t,r,\theta)}{\partial t^2}\right)\right.$$
$$\left.-\frac{2}{c^2}\left(\frac{\partial^2 f(t,r,\theta)}{\partial r^2}\right) - \frac{2}{c^4 r^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left(\frac{\partial f(t,r,\theta)}{\partial \theta}\right)^2 - \frac{4f(t,r,\theta)}{c^2 r^2}\right] = 0 \tag{2.36}$$

Simplifying equation (2.36) gives

$$G_{22} = -\frac{2r}{c^2}\frac{\partial f(t,r,\theta)}{\partial r} + \frac{1}{c^4}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left(\frac{\partial f(t,r,\theta)}{\partial \theta}\right)^2 + \frac{4r^2}{c^4}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-3}\left[\frac{\partial f(t,r,\theta)}{\partial t}\right]^2$$
$$-\frac{r^2}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left(\frac{\partial^2 f(t,r,\theta)}{\partial t^2}\right) - \frac{r^2}{c^2}\left(\frac{\partial^2 f(t,r,\theta)}{\partial r^2}\right) = 0 \tag{2.37}$$

Rearranging equation (2.37), and multiplying through with a negative sign gives

$$\frac{r^2}{c^2}\left(\frac{\partial^2 f(t,r,\theta)}{\partial r^2}\right) + \frac{2r}{c^2}\frac{\partial f(t,r,\theta)}{\partial r} - \frac{1}{c^4}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left(\frac{\partial f(t,r,\theta)}{\partial \theta}\right)^2$$
$$-\frac{4r^2}{c^4}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-3}\left[\frac{\partial f(t,r,\theta)}{\partial t}\right]^2 + \frac{r^2}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left(\frac{\partial^2 f(t,r,\theta)}{\partial t^2}\right) = 0 \tag{2.38}$$

Simplifying equation (2.38) and rearranging further gives

$$\left[r^2\left(\frac{\partial^2 f(t,r,\theta)}{\partial r^2}\right) + 2r\frac{\partial f(t,r,\theta)}{\partial r} - \frac{1}{c^2}\left(1-\frac{4f(t,r,\theta)}{c^2}+\ldots\right)\left(\frac{\partial f(t,r,\theta)}{\partial \theta}\right)^2\right.$$

$$-\frac{4r^2}{c^2}\left(1-\frac{6f(t,r,\theta)}{c^2}+...\right)\left[\frac{\partial f(t,r,\theta)}{\partial t}\right]^2+r^2\left(1-\frac{4f(t,r,\theta)}{c^2}+...\right)\left(\frac{\partial^2 f(t,r,\theta)}{\partial t^2}\right)\Bigg]=0 \tag{2.39}$$

$$\Bigg[\left(\frac{\partial^2 f(t,r,\theta)}{\partial r^2}\right)+\frac{2}{r}\frac{\partial f(t,r,\theta)}{\partial r}+\left(\frac{\partial^2 f(t,r,\theta)}{\partial t^2}\right)-\frac{1}{c^2r^2}\left(\frac{\partial f(t,r,\theta)}{\partial\theta}\right)^2+\frac{4f(t,r,\theta)}{c^4r^2}\left(\frac{\partial f(t,r,\theta)}{\partial\theta}\right)^2$$
$$-\frac{4}{c^2}\left[\frac{\partial f(t,r,\theta)}{\partial t}\right]^2+\frac{24f(t,r,\theta)}{c^4}\left[\frac{\partial f(t,r,\theta)}{\partial t}\right]^2-\frac{4f(t,r,\theta)}{c^2}\left(\frac{\partial^2 f(t,r,\theta)}{\partial t^2}\right)\Bigg]=0 \tag{2.40}$$

$$\nabla^2 f(t,r,\theta)+\left(\frac{\partial^2 f(t,r,\theta)}{\partial t^2}\right)-\frac{1}{c^2r^2}\left(\frac{\partial f(t,r,\theta)}{\partial\theta}\right)^2+\frac{4f(t,r,\theta)}{c^4r^2}\left(\frac{\partial f(t,r,\theta)}{\partial\theta}\right)^2$$
$$-\frac{4}{c^2}\left[\frac{\partial f(t,r,\theta)}{\partial t}\right]^2+\frac{24f(t,r,\theta)}{c^4}\left[\frac{\partial f(t,r,\theta)}{\partial t}\right]^2-\frac{4f(t,r,\theta)}{c^2}\left(\frac{\partial^2 f(t,r,\theta)}{\partial t^2}\right)\Bigg]=0 \tag{2.41}$$

$$\nabla^2 f(t,r,\theta)+\frac{\partial^2 f(t,r,\theta)}{\partial t^2}=0 \tag{2.42}$$

It has been shown that in the limit of c^{-2} the field equation (2.41) in the limit of weak fields reduced to:

$$\nabla^2 f(t,r,\theta)+\left(\frac{\partial^2 f(t,r,\theta)}{\partial t^2}\right)-\frac{1}{c^2r^2}\left(\frac{\partial f(t,r,\theta)}{\partial\theta}\right)^2$$
$$-\frac{4}{c^2}\left[\frac{\partial f(t,r,\theta)}{\partial t}\right]^2-\frac{4f(t,r,\theta)}{c^2}\left(\frac{\partial^2 f(t,r,\theta)}{\partial t^2}\right)\Bigg]=0 \tag{2.43}$$

Equation (2.43) can also be written as

$$\frac{\partial^2 f(t,r,\theta)}{\partial r^2}+\frac{2}{r}\frac{\partial f(t,r,\theta)}{\partial r}-\frac{1}{r^2c^2}\left(\frac{\partial f(t,r,\theta)}{\partial\theta}\right)^2-\frac{4f(t,r,\theta)}{c^2}\left(\frac{\partial^2 f(t,r,\theta)}{\partial t^2}\right)$$
$$-\frac{4}{c^2}\left(\frac{\partial f(t,r,\theta)}{\partial t}\right)^2+\frac{\partial^2 f(t,r,\theta)}{\partial t^2}=0 \tag{2.44}$$

Let us now seek a solution of equation (2.44) in the form

$$f(t,r,\theta)=\sum_{n=0}^{\infty}R_n(r)\exp n\left(t-\frac{r\theta}{c}\right) \tag{2.45}$$

where $R_n(r)$ functions of r only. By obtaining the first and second derivatives partially of equation (2.45) for $f(t,r,\theta)$; it can be shown trivially that the separate terms of the expanded equation can be shown in the equations below:

$$\frac{\partial^2 f(t,r,\theta)}{\partial r^2}=R_0^{11}(r)+\left(R_1^{11}(r)-\frac{2\theta}{c}R_1^{1}(r)-\frac{\theta^2}{c^2}R_1(r)\right)\exp\left(t-\frac{r\theta}{c}\right)$$
$$+\left(R_2^{11}(r)-\frac{2.2\theta}{c}R_2^{1}(r)-\frac{2^2\theta^2}{c^2}R_2(r)\right)\exp 2\left(t-\frac{r\theta}{c}\right)+\left(R_3^{11}(r)-\frac{2.3\theta}{c}R_3^{1}(r)-\frac{3^2\theta^2}{c^2}R_3(r)\right) \tag{2.46}$$

$$\frac{2}{r}\frac{\partial f(t,r,\theta)}{\partial r} = \frac{2}{r}R_0^{1}(r) + \frac{2}{r}R_1^{1}(r)\exp\left(t-\frac{r\theta}{c}\right) + \frac{2}{r}R_2^{1}(r)\exp 2\left(t-\frac{r\theta}{c}\right) + \frac{2}{r}R_3^{1}(r)\exp 3\left(t-\frac{r\theta}{c}\right)$$
$$-\frac{2\theta}{cr}R_1(r)\exp\left(t-\frac{r\theta}{c}\right) - \frac{2.2\theta}{cr}R_2(r)\exp 2\left(t-\frac{r\theta}{c}\right) - \frac{2.3\theta}{cr}R_3(r)\exp 3\left(t-\frac{r\theta}{c}\right) + \ldots \quad (2.47)$$

$$\frac{1}{r^2c^2}\left(\frac{\partial f(t,r,\theta)}{\partial \theta}\right)^2 = \frac{1}{c^4}R_1^{2}(r)\exp 2\left(t-\frac{r\theta}{c}\right) + \frac{2^2}{c^4}R_2^{2}(r)\exp 4\left(t-\frac{r\theta}{c}\right) + \frac{3^2}{c^4}R_3^{2}(r)\exp 6\left(t-\frac{r\theta}{c}\right) + \ldots$$
(2.48)

$$\frac{4f(t,r,\theta)}{c^2}\left(\frac{\partial^2 f(t,r,\theta)}{\partial t^2}\right) = \frac{4}{c^2}R_0(r)R_1(r)\exp\left(t-\frac{r\theta}{c}\right) + \frac{4.2^2}{c^2}R_0(r)R_2(r)\exp 2\left(t-\frac{r\theta}{c}\right)$$
$$+\frac{4.3^2}{c^2}R_0(r)R_3(r)\exp 3\left(t-\frac{r\theta}{c}\right) + \frac{4}{c^2}R_1^{2}(r)\exp 2\left(t-\frac{r\theta}{c}\right) + \frac{20}{c^2}R_1(r)R_2(r)\exp 3\left(t-\frac{r\theta}{c}\right)$$
$$+\frac{40}{c^2}R_1(r)R_3(r)\exp 4\left(t-\frac{r\theta}{c}\right) + \frac{4.2^2}{c^2}R_2^{2}(r)\exp 4\left(t-\frac{r\theta}{c}\right) + \frac{52}{c^2}R_2(r)R_3(r)\exp 6\left(t-\frac{r\theta}{c}\right) + \ldots$$
(2.49)

$$\frac{4}{c^2}\left(\frac{\partial f(t,r,\theta)}{\partial t}\right)^2 = \frac{4}{c^2}R_1^{2}(r)\exp 2\left(t-\frac{r\theta}{c}\right) + \frac{4.2^2}{c^2}R_2^{2}(r)\exp 4\left(t-\frac{r\theta}{c}\right) + \frac{4.3^2}{c^2}R_3^{2}(r)\exp 6\left(t-\frac{r\theta}{c}\right) + \ldots$$
(2.50)

$$\frac{\partial^2 f(t,r,\theta)}{\partial t^2} = R_1(r)\exp\left(t-\frac{r\theta}{c}\right) + 2^2R_2(r)\exp 2\left(t-\frac{r\theta}{c}\right) + 3^2R_3(r)\exp 3\left(t-\frac{r\theta}{c}\right) + \ldots$$
(2.51)

Comparing coefficients of $\exp(0)$

$$R_0^{11}(r) + \frac{2}{r}R_0^{1}(r) = 0 \quad (2.52)$$

Solving equation (2.52) to obtain the auxiliary solution for the second order partial differential equation gives:

$$R_0(r) = -\frac{2}{r} \quad (2.53)$$

But according to Newton's dynamical theory, Newton's gravitational scalar potential exterior to a distribution of mass or pressure is given by

$$f(r) = -\frac{GM_0}{r} \quad (2.54)$$

Comparing equation (2.53) with Newton's gravitational scalar potential (2.54) we can choose the most convenient astrophysical solution for (2.52) as:

$$R_0(r) \approx -\frac{k}{r} \quad (2.55)$$

where $k = GM_0$; deducing from Schwarzschild's metric and Newton's dynamical theory of gravitation, G is the universal gravitational constant and M_0 is the total mass of the spherical body.

Equating coefficients of $\exp\left(t - \frac{r\theta}{c}\right)$ yields

$$R_1^{11}(r) + 2\left[\frac{1}{r} - \frac{\theta}{c}\right]R_1^{1}(r) - \frac{\theta}{c}\left[\theta + \frac{2}{r} + \frac{4}{c}R_0(r)\right]R_1(r) = 0 \qquad (2.56)$$

This is the exact differential equation for R_1 and it determines R_1 in terms of R_0. Thus, the solution assumes an exact wave equation, which in the order of c^0 reduces to

$$f(t,r,\theta) \approx -\frac{k}{r}\exp\left(t - \frac{r\theta}{c}\right) \qquad (2.57)$$

Interestingly and remarkably, we obtain an arbitrary function which is a function of radial distance, polar angle and time equal to Newton's scalar potential, hence our obtained result could be apply in all the applications of Newton's scalar potential with a much wider application such as the study of coupling effects of electromagnetism, weak field approximation.

CONCLUSION

The result obtained in this article is applicable to all 2-D dynamical physical systems rotating about a fixed point or a phenomenon originating from a fixed point (Howusu, 2010).

Furthermore, the obtained results in this article differ from Chifu and Lumbi (2008); Chifu (2009) and Sarki, *et. al.*(2018) . Chifu and Lumbi (2008) studied a hypothetical systems which varies with azimuthal angle only, whereas Chifu (2009) is for static homogenous oblate spheroidal systems, and Sarki, *et. al.*(2018) is for a static astrophysical systems which varies with radial distance and azimuthal angle only.

Instructively, our single dependent function $f(t,r,\theta)$ which is our physically and mathematically most satisfactory solution contains unknown post Newtonian terms or pure Einsteinian gravitational terms in order of c^0 and c^{-2} . Hence, this research work has shown that the Exterior EGGFE can be obtained as a generalization or completion of Newton's dynamical gravitational field equations.

Interestingly, we discover that the solution obtained, that is equation (2.56) has a particular link to the pure Newtonian gravitational scalar potential for the gravitational field and hence put Einstein's geometrical gravitational field on the same level with the Newtonian dynamical theory of gravitation as obtained by Chifu (2012); Chifu (2009) and Chifu *et al* (2009).

The gravitational scalar potential obtained in this research work can be applied in:

i. The study of rotating astrophysical bodies within a spherical geometry whose tensor field varies with time, radial distance and polar angle. Example of such bodies are stars such as Neutron star, Wolf-Rayet, e.t.c.
ii. The study of astrophysical phenomenon such as gravitational red-shift by the sun, time dilation, length contraction, motion of particles and photons.
iii. The study of gravitomagnetic and gravitoelectric coupling, just to mention but a few.

Thus we have completely obtained the solution of EGGFE exterior to homogeneous spherical bodies whose tensor fields varies with time, radial distance and polar angle.The solutions of the Einstein's field equations are metrics of space time. These metrics describe the structure of the space-time including the inertial motion of objects in the space-time. As the field equations are non-linear, they cannot always be completely solved (that is without approximation).

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