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Seven-Step Hybrid Block Extended second order Derivative backward Differentiation Formula

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Abstract

Second derivative hybrid block backward differentiation formula of uniform order 10 was presented for the numerical solution of stiff initial value problems. The schemes with one off-grid point at first derivative have been derived for solving Stiff Ordinary Differential Equations (SODEs) using interpolation and collocation procedures. In addition, the order and error constant have been investigated. The convergence analysis reveals that the methods $k = 7$ and their block forms are A-stable.The newly constructed method are applied to stiff ordinary differential equations arising from real life situation has shown that it is efficient and compare favourably with other existing second order derivative methods and the exact solutions.

Keywords: Hybrid, Block Method, Backward Differentiation Formula, Second Derivative, Stiff ODEs

INTRODUCTION

An ordinary differential equation problem is stiff if the solution being sought is vary slowly, but there are nearby solutions that vary rapidly. In some cases, the differential equations could be solved analytically while others may be too complicated to use analytic methods of solutions. Numerical methods are often used to give approximate solutions to these differential equations.

Continuous collocation and interpolation technique are now widely used for the derivation of linear multistep methods (LMMs) and hybrid block methods. Several continuous LMMs have been derived using different techniques and approaches. Alabi (2008) derived continuous solvers of IVPs using Chebyshev polynomial in a linear multistep collocation technique. Okunugu and Ehigie (2009) derived two-step continuous and discrete LMMs using power series as basis functions.

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Mohammed (2011) derived a linear multistep method with continuous coefficients and used it to obtain multiple finite difference methods which were directly applied to solve first-order ODEs.

Odekunle *et al* (2012) developed a continuous linear multistep method using interpolation and collocation for the solution of first-order ODE with constants step size. Adesanya *et al* (2012) considered the method of collocation of the differential system and interpolation of the approximate solution to generate a continuous of LMMs which is solved for the independent solution to yield a continuous block method. James *et al* (2013) proposed a continuous block method for the solution of second order IVPs with constant step size the method was developed by interpolation and collocation of power series approximate solution. Anake (2011) developed a new class of continuous implicit hybrid one-step methods capable of solving IVPs of general second order ODEs using the interpolation and collocation techniques of the power series approximate solution. Ehigie *et al* (2010) proposed a two-step continuous multistep method of hybrid type for the direct integration of second order ODEs in a multistep collocation technique. Although, a very wide variety of numerical methods have been proposed, the number of methods with high order and good stability properties remains relatively small. We develop a continuous hybrid second derivative block backward differentiation formula based on interpolation and collocation for the solution of stiff ordinary differential equations with constants step size.

$$y(x) = \sum_{j=0}^{r-1} \alpha_j(x) y_{n+j} + h \sum_{j=0}^{s-1} \beta_j(x) f_{n+j} + h^2 \sum_{j=0}^{t-1} \gamma_j(x) g_{n+j} \qquad (1)$$

where s, t the number of distinct collocation points and r the number of interpolation points. Moreover,α_j, β_j andγ_j are expressed as continuous functions of x and are at least differentiable once Awoyemi*et al* (2014).

Notations used:
HBESODBDF: hybrid block extended second order derivative backward differentiation formula

FORMULATION OF SEVEN-STEP HBESODBDF METHOD
The concept is to approximate the analytic solution $y(x)$ in the partition $\pi[a,b]$=$[a = x_0 < x_1 < \cdots < x_n = b]$ of the integration interval$[a,b]$ by a power series polynomial of the form.

$$y(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_p x^p = \sum_{j=0}^{p} a x^j \qquad (2)$$

where $p = r + s + t - 1$
which is twice-continuously differentiable function of $y(x)$.

$$y'^{(x)} = \sum_{j=0}^{r+s+t-1} j a_j x^{j-1} \qquad (3)$$

Now, we differentiate the function $y(x)$ twice

$$y''^{(x)} = \sum_{j=0}^{r+s+t-1} j(1-j) a_j x^{j-2} \qquad (4)$$

where $x \in [a,b]$, the $a's$are real unknown parameters to be determined and $r + s + t$ is the sum of the number of collocation and interpolation points.

SPECIFICATION of SEVEN-STEP HBESODBDF METHOD

Considering equation (2), (3) and (4) we propose a continuous scheme for $seven-step$ of hybrid block extended second derivative backward differentiation formula.

$$y(x) = \sum_{j=0}^{r-1}\alpha_j(x)y_{n+j} + \sum_{j=0}^{s-1}h\beta_{k-1}(x)f_{n+k-1} + h\beta_{\frac{2k-1}{2}}f_{n+\frac{2k-1}{2}} + h\beta_k(x)f_{n+k} + \sum_{j=0}^{t-1}h^2\gamma_{k-1}(x)g_{n+k-1} + h^2\gamma_k(x)g_{n+k} \quad (5)$$

In this case $k = 7$, $r = 3$,$S = 2$, $t = 7$ and (5) becomes

$$y(x) = \alpha_0(x)y_n + \alpha_1(x)y_{n+1} + \alpha_2(x)y_{n+2} + \alpha_3(x)y_{n+3} + \alpha_4(x)y_{n+4} + \alpha_5(x)y_{n+5} + \alpha_6(x)f_{n+6} + h[\beta_0(x)f_n + \beta_1(x)f_{n+1} + \beta_2(x)f_{n+2} + \beta_3(x)f_{n+3} + \beta_4(x)f_{n+4} + \beta_5(x)f_{n+5} + \beta_6(x)f_{n+6} + \beta_{\frac{13}{2}}(x)f_{n+\frac{13}{2}} + \beta_7(x)f_{n+7}] + h^2[\gamma_0(x)g_n + \gamma_1(x)g_{n+1} + \gamma_2(x)g_{n+2} + \gamma_3(x)g_{n+3} + \gamma_4(x)g_{n+4} + \gamma_5(x)g_{n+5} + \gamma_6(x)g_{n+6} + \gamma_7(x)g_{n+7}] \quad (6)$$

and interpolating (2) at $x = x_n, x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4}, x_{n+5}, x_{n+6}$ and collocating (3) at $x = x_{n+6}, x_{n+\frac{13}{2}}, x_{n+7}$ to give the systems of equation.

$$y(x) = \sum_{j=0}^{6} a_j x^j = y_n \quad (7)$$

$$y'(x) = \sum_{j=1}^{6} j a_j x^{j-1} = f_{n+j} \quad (8)$$

$$y''(x) = \sum_{j=2}^{6} j(j-1) a_j x^{j-2} = f_{n+j} \quad (9)$$

Expressing the system of equations (7) -(9) in the form $AX = Y$ as:

$$\begin{bmatrix}
1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 & x_n^8 & x_n^9 & x_n^{10} & x_n^{11} \\
1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 & x_{n+1}^7 & x_{n+1}^8 & x_{n+1}^9 & x_{n+1}^{10} & x_{n+1}^{11} \\
1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 & x_{n+2}^5 & x_{n+2}^6 & x_{n+2}^7 & x_{n+2}^8 & x_{n+2}^9 & x_{n+2}^{10} & x_{n+2}^{11} \\
1 & x_{n+3} & x_{n+3}^2 & x_{n+3}^3 & x_{n+3}^4 & x_{n+3}^5 & x_{n+3}^6 & x_{n+3}^7 & x_{n+3}^8 & x_{n+3}^9 & x_{n+3}^{10} & x_{n+3}^{11} \\
1 & x_{n+4} & x_{n+4}^2 & x_{n+4}^3 & x_{n+4}^4 & x_{n+4}^5 & x_{n+4}^6 & x_{n+4}^7 & x_{n+4}^8 & x_{n+4}^9 & x_{n+4}^{10} & x_{n+4}^{11} \\
1 & x_{n+5} & x_{n+5}^2 & x_{n+5}^3 & x_{n+5}^4 & x_{n+5}^5 & x_{n+5}^6 & x_{n+5}^7 & x_{n+5}^8 & x_{n+5}^9 & x_{n+5}^{10} & x_{n+5}^{11} \\
1 & x_{n+6} & x_{n+6}^2 & x_{n+6}^3 & x_{n+6}^4 & x_{n+6}^5 & x_{n+6}^6 & x_{n+6}^7 & x_{n+6}^8 & x_{n+6}^9 & x_{n+6}^{10} & x_{n+6}^{11} \\
0 & 1 & 2x_{n+6} & 3x_{n+6}^2 & 4x_{n+6}^3 & 5x_{n+6}^4 & 6x_{n+6}^5 & 7x_{n+6}^6 & 8x_{n+6}^7 & 9x_{n+6}^8 & 10x_{n+6}^9 & 11x_{n+6}^{10} \\
0 & 1 & 2x_{n+\frac{13}{2}} & 3x_{n+\frac{13}{2}}^2 & 4x_{n+\frac{13}{2}}^3 & 5x_{n+\frac{13}{2}}^3 & 6x_{n+\frac{13}{2}}^5 & 7x_{n+\frac{13}{2}}^6 & 8x_{n+\frac{13}{2}}^7 & 9x_{n+\frac{13}{2}}^8 & 10x_{n+\frac{13}{2}}^9 & 11x_{n+\frac{13}{2}}^{10} \\
0 & 1 & 2x_{n+7} & 3x_{n+7}^2 & 4x_{n+7}^3 & 5x_{n+7}^3 & 6x_{n+7}^5 & 7x_{n+7}^6 & 8x_{n+7}^7 & 9x_{n+7}^8 & 10x_{n+7}^9 & 11x_{n+7}^{10} \\
0 & 0 & 2 & 6x_{n+6} & 12x_{n+6}^2 & 15x_{n+6}^3 & 30x_{n+6}^3 & 42x_{n+6}^5 & 56x_{n+6}^7 & 72x_{n+6}^7 & 90x_{n+6}^6 & 110x_{n+6}^9 \\
0 & 0 & 2 & 6x_{n+7} & 12x_{n+7}^2 & 15x_{n+7}^3 & 30x_{n+7}^3 & 42x_{n+7}^5 & 56x_{n+7}^7 & 72x_{n+7}^7 & 90x_{n+7}^6 & 110x_{n+7}^9
\end{bmatrix}
\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \\ a_{10} \\ a_{11} \end{bmatrix}
=
\begin{bmatrix} y_n \\ y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \\ y_{n+5} \\ y_{n+6} \\ f_{n+6} \\ f_{n+\frac{13}{2}} \\ f_{n+7} \\ g_{n+6} \\ g_{n+7} \end{bmatrix} \quad (10)$$

Using Maple software and inverting the matrix in (10) is yields the elements of the matrix A^{-1}`.

The columns of A^{-1}`give the continuous coefficients of (6) as:

$$\begin{aligned}
\alpha_0(x) = 1 &- \frac{6040393412599x}{1879532849820h} + \frac{2048598444193x^2}{469883212455h^2} - \frac{56666432755721x^3}{16915795648380h^3} \\
&+ \frac{166242263558761x^4}{101494773890280h^4} - \frac{437975009777423x^5}{811958191122240h^5} + \frac{11080135063861x^6}{90217576791360h^6} \\
&- \frac{175227777949x^7}{9021757679136h^7} + \frac{283972892777x^8}{135326365187040h^8} - \frac{40040230493x^9}{270652730374080h^9} \\
&+ \frac{4985373157x^{10}}{811958191122240h^{10}} - \frac{11557589x^{11}}{101494773890280h^{11}}
\end{aligned}$$

$$\begin{aligned}
\alpha_1(x) = &\frac{6040393412599x}{1879532849820h} - \frac{4052403986988721x^2}{133133576862250h^2} + \frac{3928342347441207x^3}{133133576862250h^3} \\
&- \frac{9916622961685558x^4}{599101095880125h^4} + \frac{42904219787970043x^5}{7189213150561500h^5} \\
&- \frac{27768159122782711x^6}{19171235068164000h^6} + \frac{2301851593181653x^7}{9585617534082000h^7} \\
&- \frac{128920461211681x^8}{4792808767041000h^8} + \frac{2077975484429x^9}{1065068614898000h^9} \\
&- \frac{18685079441x^{10}}{225543941978400h^{10}} - \frac{22508952547x^{11}}{14378426301123000h^{11}}
\end{aligned}$$

$$\begin{aligned}
\alpha_2(x) = &-\frac{6837782666715x}{146906015848h} + \frac{74223006411477x^2}{587624063392h^2} - \frac{164257224123747x^3}{1175248126784h^3} \\
&+ \frac{606631947310451x^4}{7051488760704h^4} - \frac{1405314013540099x^5}{42308932564224h^5} + \frac{359428676241743x^6}{42308932564224h^6} \\
&- \frac{31049441947063x^7}{21154466282112h^7} + \frac{1797662130605x^8}{10577233141056h^8} \\
&- \frac{535924815683x^9}{42308932564224h^9} + \frac{1369221811x^{10}}{2488760739072h^{10}} - \frac{3502385x^{11}}{330538535658h^{11}}
\end{aligned}$$

$$\alpha_3(x) = \frac{230523157357220x}{1597602922347h} - \frac{221326459001263x^2}{532534307449h^2} + \frac{2350862427100247x^3}{4792808767041h^3} - \frac{13794649287550432x^4}{43135278903369h^4} + \frac{11188738268091743x^5}{86270557806738h^5} - \frac{2651711513278865x^6}{76684940272656h^6} + \frac{3295646999079x^7}{532534307449h^7} - \frac{84942518883997x^8}{115027410408984h^8} + \frac{3248092601459x^9}{57513705204492h^9} - \frac{101825920369x^{10}}{40597909556112h^{10}} + \frac{8498097839x^{11}}{172541115613476h^{11}}$$

$$\alpha_4(x) = -\frac{928560969765975x}{2130137229796h} + \frac{687956890973065x^2}{532534307449h^2} - \frac{839930344116147x^3}{532534307449h^3} + \frac{81713559494845805x^4}{76684940272656h^4} - \frac{411640370121028715x^5}{920219283271872h^5} + \frac{37783890196273927x^6}{306739761090624h^6} - \frac{3482472195659983x^7}{153369880545312h^7} + \frac{426836821452263x^8}{153369880545312h^8} - \frac{7431943475369x^9}{34082195676736h^9} + \frac{178602277447x^{10}}{18043515358272h^{10}} - \frac{22804108043x^{11}}{115027410408984h^{11}}$$

$$\alpha_5(x) = \frac{5073205624157034x}{2662671537245h} - \frac{30574015613842449x^2}{5325343074490h^2} + \frac{38106324681935163x^3}{5325343074490h^3} - \frac{39523412309245102x^4}{7988014611735h^4} + \frac{204042282154502089x^5}{95856175340820h^5} - \frac{460774750721268871x^6}{766849402726560h^6} + \frac{8703472262733655x^7}{76684940272656h^7} - \frac{682613284856173x^8}{47928087670410h^8} + \frac{43595727098863x^9}{383424701363280h^9} - \frac{2387124030089x^{10}}{45108788395680h^{10}} + \frac{207298720337x^{11}}{191712350681640h^{11}}$$

$$\alpha_6(x) = -\frac{504099468152041307x}{319520584469400h} + \frac{3044873281075893623 3x^2}{6390411689388000h^2} - \frac{685024437285456259223x^3}{115027410408984000h^3} + \frac{285103457708262327619 1x^4}{690164462453904000h^4} - \frac{2461412803123504996381x^5}{1380328924907808000h^5} + \frac{774668081646719203 13x^6}{153369880545312000h^6} - \frac{7341180482345822849x^7}{76684940272656000h^7} + \frac{1386389924889505469x^8}{115027410408984000h^8} - \frac{148589937019902901x^9}{153369880545312000h^9} + \frac{731664415217177x^{10}}{16239163822444800h^{10}} - \frac{19910374829633x^{11}}{21567639451684500h^{11}}$$

$$\beta_6(x) = -\frac{354413161477739x}{5325343074490} + \frac{26432451553362841x^2}{106506861489800h} - \frac{741595748974323871x^3}{1917123506816400h^2} + \frac{1275087010915177369x^4}{3834247013632800h^3} - \frac{1346668666155108979x^5}{7668494027265600h^4} + \frac{51061186969083767x^6}{852054891918400h^5} - \frac{17220844046192773\,x^7}{1278082337877600h^6} + \frac{1267272580883071x^8}{639041168938800h^7} - \frac{1409767676638931x^9}{76684940272656000Sh^9} + \frac{878623287703x^{10}}{90217576791360h^{10}} - \frac{26934533447x^{11}}{119820219176025h^{11}}$$

$$\beta_{\frac{13}{2}}(x) = \frac{992589290864640x}{532534307449} - \frac{3038689938751488x^2}{532534307449h} + \frac{386480785754112x^3}{532534307449h^2} - \frac{27360900989747720x^4}{532534307449h^3} + \frac{3626211223281664x^5}{1597602922347h^4} - \frac{1053422542527488x^6}{1597602922347h^5} - \frac{205107288028160\,x^7}{1597602922347h^6} - \frac{26560959703040x^8}{1597602922347h^7} + \frac{2197931841536x^9}{1597602922347h^8} - \frac{6192937984x^{10}}{93976642491h^9} + \frac{2222347264x^{11}}{1597602922347h^{10}}$$

$$\beta_7(x) = -\frac{248536898681760x}{532534307449} + \frac{763001312465112x^2}{532534307449h} - \frac{973842408895932x^3}{532534307449h^2} + \frac{2077533637403848x^4}{1597602922347h^3} - \frac{2767262917876444x^5}{4792808767041h^4} + \frac{538963087891585x^6}{3195205844694h^5} - \frac{211188256090405\,x^7}{6390411689388h^6} + \frac{27533996390033x^8}{6390411689388h^7} - \frac{765020377001x^9}{2130137229796h^8} + \frac{6516762533x^{10}}{375906569964h^9} - \frac{1768258022x^{11}}{4792808767041h^{10}}$$

$$\gamma_6(x) = -\frac{543896538211299xh}{532534307449} + \frac{33124571825047381x^2}{10650686148980} - \frac{251031635363669267x^3}{63904116893880h} + \frac{105777854378369060 9x^4}{383424701363280h^2} - \frac{925963117414499569x^5}{766849402726560h^3} + \frac{29582629650654857x^6}{85205486191840h^4} - \frac{2848077765284381\,x^7}{42602744595920h^5} + \frac{136678946839589x^8}{15976029223470h^6} - \frac{59576045739849x^9}{85205486191840h^7} + \frac{298280738849x^{10}}{9021757679136h^8} - \frac{33009263123x^{11}}{47928087670410h^9}$$

$$\gamma_7(x) = \frac{41692321059480xh}{532534307449} - \frac{128150608331646x^2}{532534307449} + \frac{163821849939630x^3}{532534307449h} - \frac{116719856198376x^4}{532534307449h^2} + \frac{51939996963383x^5}{532534307449h^3} - \frac{121704413471641x^6}{4260274459592h^4} + \frac{2989162874887\,x^7}{532534307449h^5} - \frac{1563870566441x^8}{21301372297966h^6} + \frac{65407846229x^9}{1065068614898h^7} - \frac{745776053x^{10}}{250604379976h^8} + \frac{67738411x^{11}}{1065068614898h^9}$$

Evaluating (6) at the following points $x_n, x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4}, x_{n+5}, x_{n+13/2}$ *and* x_{n+7} yields the following discrete methods which constitute the new seven step block method.

$$y_n - \frac{1736744565852309}{248758382509150}y_{n+1} + \frac{4612429684141785}{159205364805856}y_{n+2} - \frac{474270983574135}{4975167650183}y_{n+3} + \frac{77589123042075}{261850928957}y_{n+4} - \frac{13103149548789621}{9950335300366}y_{n+5} + \frac{434981897296556231 9}{3980134120146400}y_{n+6}$$
$$= -\frac{3h}{199006706007320}\left[3776064507623263f_{n+6} - 86819712535756800f_{n+\frac{13}{2}} + 21800037499003200f_{n+7}\right] + \frac{3h^2}{139304694205124}[5325343074490g_n - 331245718250473 81g_{n+6} + 2563012166632920g_{n+7}]$$

$$y_n + \frac{49075323184639}{10576115928375}y_{n+1} - \frac{3909160078455}{112811903236}y_{n+2} + \frac{3230385802860}{28202975809}y_{n+3} - \frac{200190712038185}{592262491989}y_{n+4} + \frac{285771785959677}{197420830663}y_{n+5} - \frac{117963356313224339}{98710415331500}y_{n+6}$$
$$= \frac{h}{4935520766575}[166397889103509f_{n+6} - 6834212315136000f_{n+\frac{13}{2}} + 1704059899541500f_{n+7}] + \frac{2h^2}{987104153315}[1065068614898g_{n+1} + 377419448503431g_{n+6} - 28532857129550g_{n+7}]$$

$$y_n - \frac{285307179863964}{8444016421525}y_{n+1} + \frac{8523684438015}{2702085254888}y_{n+2} + \frac{91872298627960}{337760656861}y_{n+3} - \frac{319756201989735}{337760656861}y_{n+4} + \frac{1328692387575684}{337760656861}y_{n+5} - \frac{218159883646192663}{67552131372200}y_{n+6}$$
$$= \frac{9h}{3377606568610}\left[1116591943517f_{n+6} - 1350622517657600f_{n+\frac{13}{2}} + 333923870528000f_{n+7}\right] - \frac{9h^2}{337760656861}[1597602922347g_{n+2} - 757745772097 42g_{n+6} + 5570577683568g_{n+7}]$$

$$y_n - \frac{840719081967}{37225934875}y_{n+1} + \frac{459080773587}{1191229916}y_{n+2} - \frac{122478484400}{297807479}y_{n+3} - \frac{374935973163}{297807479}y_{n+4} + \frac{2238402684555}{297807479}y_{n+5} - \frac{924665237606007}{148903739500}y_{n+6}$$
$$= -\frac{3h}{7445186975}\left[347089383461f_{n+6} + 16626858393600f_{n+\frac{13}{2}} - 4071163275300f_{n+7}\right] + \frac{6h^2}{1489037395}[91816259905g_{n+3} + 948052579409g_{n+6} - 67620255705g_{n+7}]$$

$$y_n - \frac{840719081967}{37225934875}y_{n+1} + \frac{459080773587}{1191229916}y_{n+2} - \frac{122478484400}{297807479}y_{n+3} - \frac{374935973163}{297807479}y_{n+4} + \frac{2238402684555}{297807479}y_{n+5} - \frac{924665237606007}{148903739500}y_{n+6} = -\frac{3h}{74455186975}\left[347089383461f_{n+6} + 16626858393600f_{n+\frac{13}{2}} - 4071163275300f_{n+7}\right] + \frac{6h^2}{1489037395}[91816259905g_{n+3} + 948052579409g_{n+6} - 67620255705g_{n+7}]$$

$$y_n - \frac{12835025643453}{731560018675}y_{n+1} + \frac{19166858741355}{117049602988}y_{n+2} - \frac{35612575945940}{29262400747}y_{n+3} + \frac{339705742234425}{29262400747}y_{n+4} - \frac{64843273426407}{29262400747}y_{n+5} - \frac{24355746892242563}{2926240074700}y_{n+6} = \frac{9h}{146312003735}\left[30860366345917f_{n+6} - 417799787315200f_{n+\frac{13}{2}} + 95547248068300f_{n+7}\right] + \frac{18h^2}{29262400747}[15976029223470g_{n+5} + 25744881427247g_{n+6} - 1542524765490g_{n+7}]$$

$$y_{n+\frac{13}{2}} - \frac{780582495}{2052951080763392}y_n + \frac{542209728879}{87250420932444160}y_{n+1} - \frac{249623499125}{4813816327307264}y_{n+2} + \frac{2740997763075}{8725042093244416}y_{n+3} - \frac{63329157580689}{34900168372977664}y_{n+4} + \frac{305254689835095}{17450084186488832}y_{n+5} - \frac{709134128594798907}{698003367459553280}y_{n+6} = \frac{9009h}{34900168372977664}\left[1190799022413f_{n+6} + 736660848640f_{n+\frac{13}{2}} - 47165118000f_{n+7}\right] + \frac{135270135h^2}{17450084186488832}[6012001g_{n+6} + 216540g_{n+7}]$$

$$y_{n+7} + \frac{17900}{313225547497}y_n - \frac{4855508}{532534307449}y_{n+1} + \frac{1349019}{18363251981}y_{n+2} - \frac{224107625}{532534307449}y_{n+3} + \frac{1166714500}{532534307449}y_{n+4} - \frac{8646378300}{532534307449}y_{n+5} - \frac{524865106367}{532534307449}y_{n+6} = \frac{1260h}{532534307449}\left[9333867f_{n+6} + 240844800f_{n+\frac{13}{2}} + 93900287f_{n+7}\right] + \frac{88200h^2}{532534307449}[14147g_{n+6} - 88069g_{n+7}] \quad (11)$$

ANALYSIS OF THE NEW METHODS

In this section, the analysis of the newly constructed method (SEVEN-STEP HBESODBDF)is carried out by analyzing the order, error constant consistency, convergence and zero stability.

we defined the local truncation error associated with (1) to be linear difference operator (Lambert,1973)

$$[y(x);h] = \sum_{j=0}^{k} \alpha_j y_{n+j} - h\beta_k f_{n+k} - h^2\gamma_k g_{n+k} \quad . \quad (12)$$

Assuming that $y(x)$ is sufficiently differentiable, we can expand the terms in (12) as a Taylor series and comparing the coefficients of h gives

$$L[y(x);h] = c_0 y(x) + c_1 h y'^{(x)} + c_2 h^2 y''^{(x)} + \cdots + c_p h^p y^p(x) + \cdots \quad (13)$$

where the constants $C_p, p = 0,1,2 \ldots.,j = 1,2,\ldots k$ are given as follows:

$$C_0 = \sum_{j=0}^{k} \alpha_j$$

$$C_1 = \sum_{j=1}^{k} j\alpha_j - \sum_{j=0}^{k} \beta_j$$

. . .

. . .

$$C_q = \frac{1}{q!}\sum_{j=0}^{k} j^q \alpha_j - \frac{1}{(q-1)!}\sum_{j=1}^{k} j^{q-1}\beta_j - \frac{1}{(q-2)!}\sum_{j=1}^{k} j^{q-2}\lambda_j \qquad (14)$$

where

$C_0 = C_1 = C_2 \dots C_p = C_{p+1} = 0$ and $C_{p+2} \neq 0$. Therefore,C_{p+2} is the error constant and $C_{p+2}h^{p+2}y^{(p+2)}(x_n)$ is the principal local truncation error at apoint x_n. It was established from the evaluation of both hybrid block methods have order and error constants.

Using the concept above, the hybrid block methods are obtained with the help of MAPLE 18 SOFTWARE have the following uniform order and error constants

Table 1 ORDER AND ERROR CONSTANTS OF THE SEVEN -STEP

Method	Order	Error Constant
y_n	10	$-\frac{4965391873495939}{367764392701527360}$
y_{n+1}	10	$\frac{68764967533447}{5472505425978360}$
y_{n+2}	10	$\frac{88165482138851}{3120908469395640}$
y_{n+3}	10	$\frac{306111153059}{6879352764900}$
y_{n+4}	10	$\frac{173189079851}{2779964337920}$
y_{n+5}	10	$\frac{22694743922483}{270384582902280}$
$y_{n+\frac{13}{2}}$	10	$-\frac{97959014153}{2233610775870570496}$
y_{n+7}	10	$\frac{1317610}{17573632145817}$

ZERO STABILITY OF THE EBHBDF

The zero stability of the new methods are determined using the approach of Ehigie *et al.*(2014) in which he expressed the block method as

$$\begin{bmatrix} a_{11} & \cdot & \cdot & \cdot & a_{1n} \\ \cdot & & \cdot & & \cdot \\ a_{1n} & \cdot & \cdot & \cdot & a_{nn} \end{bmatrix}\begin{bmatrix} y_{n+1} \\ \cdot \\ y_{n+n} \end{bmatrix} = \begin{bmatrix} b_{11} & \cdot & \cdot & \cdot & b_{1n} \\ \cdot & & \cdot & & \cdot \\ b_{1n} & \cdot & \cdot & \cdot & b_{nn} \end{bmatrix}\begin{bmatrix} y_{n-1} \\ \cdot \\ y_{n} \end{bmatrix} + \begin{bmatrix} c_{11} & \cdot & \cdot & \cdot & c_{1n} \\ \cdot & & \cdot & & \cdot \\ c_{1n} & \cdot & \cdot & \cdot & c_{nn} \end{bmatrix}\begin{bmatrix} f_{n+1} \\ \cdot \\ f_{n+n} \end{bmatrix} + \begin{bmatrix} d_{11} & \cdot & \cdot & \cdot & d_{1n} \\ \cdot & & \cdot & & \cdot \\ d_{1n} & \cdot & \cdot & \cdot & d_{nn} \end{bmatrix}\begin{bmatrix} g_{n+1} \\ \cdot \\ g_{n+n} \end{bmatrix}$$

where $A = \begin{bmatrix} a_{11} & \cdot & \cdot & \cdot & a_{1n} \\ \cdot & & \cdot & & \cdot \\ a_{1n} & \cdot & \cdot & \cdot & a_{nn} \end{bmatrix}, B = \begin{bmatrix} b_{11} & \cdot & \cdot & \cdot & b_{1n} \\ \cdot & & \cdot & & \cdot \\ b_{1n} & \cdot & \cdot & \cdot & b_{nn} \end{bmatrix}, C = \begin{bmatrix} c_{11} & \cdot & \cdot & \cdot & c_{1n} \\ \cdot & & \cdot & & \cdot \\ c_{1n} & \cdot & \cdot & \cdot & c_{nn} \end{bmatrix}$

$, D = \begin{bmatrix} d_{11} & \cdot & \cdot & \cdot & d_{1n} \\ \cdot & & \cdot & & \cdot \\ d_{1n} & \cdot & \cdot & \cdot & d_{nn} \end{bmatrix}$ (15)

and the stability polynomial given by

$$det[r(A - Cz - Dz^2 - B)] = 0 \qquad (16)$$

Substituting (15) into (16) for r as $z \to 0$ produce the characteristic equation and is solved for roots of r.

ZERO STABILITY OF THE SEVEN-STEP EBHBDF

The seven-step EBHBDF method is expressed in the form of (15) using Maple yields the stability polynomial of the method as

$$\rho(r) = \det(rA - B)$$

$$det\left[r \begin{bmatrix} \frac{-1736744565862309}{248758382509150} & \frac{4612429684141785}{159205364805856} & \frac{-474270983574135}{4975167650183} & \frac{77589123042075}{261850928957} & \frac{-13103149548789621}{99503353300366} & \frac{4349818972965562319}{3980134120146400} & 0 & 0 \\ \frac{49075323184639}{10576115928375} & \frac{-3909160078455}{112811903236} & \frac{3230385802860}{28202975809} & \frac{-200190712038185}{592262491989} & \frac{285771785959677}{197420830663} & \frac{-117963356313224339}{98710415331500} & 0 & 0 \\ \frac{285307179863964}{8444016421525} & \frac{8523684438015}{2702085254888} & \frac{91872298627960}{3377606576861} & \frac{-319756201989735}{3377606576861} & \frac{1328692387575684}{3377606576861} & \frac{-218159883646192663}{67552131372200} & 0 & 0 \\ \frac{-8407619081967}{37225934875} & \frac{459080773587}{1191229916} & \frac{-122478484000}{297807479} & \frac{-374935973163}{297807479} & \frac{2238042684555}{297807479} & \frac{-924665237606007}{148903739500} & 0 & 0 \\ \frac{-108572791721}{5641161400} & \frac{110375210025}{515763328} & \frac{-282903165835}{112823228} & \frac{347286493405}{112823228} & \frac{2190449526759}{225646456} & \frac{945355228428839}{90258582400} & 0 & 0 \\ \frac{-12835025643453}{731560018675} & \frac{19166858741355}{117049602988} & \frac{-35612575945940}{29262400747} & \frac{339705742234425}{29262400747} & \frac{64843273426407}{29262400747} & \frac{24355746892242563}{2926240074700} & 0 & 0 \\ \frac{542209728879}{87250420932444160} & \frac{-249623499125}{4813816327307264} & \frac{2740997763075}{8725042093244416} & \frac{63329157580689}{34900168372977664} & \frac{305254689835095}{17450084186488832} & \frac{-709134128594798907}{698003367459553280} & 1 & 0 \\ \frac{-4855508}{532534307449} & \frac{1349019}{18363251981} & \frac{-224107625}{532534307449} & \frac{1166714500}{532534307449} & \frac{-8646378300}{532534307449} & \frac{-524865106367}{532534407449} & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{780582495}{2052951080763392} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{17900}{31325547497} \end{bmatrix} \right]$$

$$det \begin{bmatrix} \frac{-1736744565862309}{248758382509150}r & \frac{4612429684141785}{159205364805856}r & \frac{-474270983574135}{4975167650183}r & \frac{77589123042075}{261850928957}r & \frac{-13103149548789621}{99503353300366}r & \frac{4349818972965562319}{3980134120146400}r & 0 & 1 \\ \frac{49075323184639}{10576115928375}r & \frac{-3909160078455}{112811903236}r & \frac{3230385802860}{28202975809}r & \frac{-200190712038185}{592262491989}r & \frac{285771785959677}{197420830663}r & \frac{-117963356313224339}{98710415331500}r & 0 & 1 \\ \frac{285307179863964}{8444016421525}r & \frac{8523684438015}{2702085254888}r & \frac{91872298627960}{3377606576861}r & \frac{-319756201989735}{3377606576861}r & \frac{1328692387575684}{3377606576861}r & \frac{-218159883646192663}{67552131372200}r & 0 & 1 \\ \frac{-8407619081967}{37225934875}r & \frac{459080773587}{1191229916}r & \frac{-122478484000}{297807479}r & \frac{-374935973163}{297807479}r & \frac{2238042684555}{297807479}r & \frac{-924665237606007}{148903739500}r & 0 & 1 \\ \frac{-108572791721}{5641161400}r & \frac{110375210025}{515763328}r & \frac{-282903165835}{112823228}r & \frac{347286493405}{112823228}r & \frac{2190449526759}{225646456}r & \frac{945355228428839}{90258582400}r & 0 & 1 \\ \frac{-12835025643453}{731560018675}r & \frac{19166858741355}{117049602988}r & \frac{-35612575945940}{29262400747}r & \frac{-35612575945940}{29262400747}r & \frac{2190449526759}{225646456}r & \frac{945355228428839}{90258582400}r & 0 & 1 \\ \frac{542209728879}{87250420932444160}r & \frac{-249623499125}{4813816327307264}r & \frac{2740997763075}{8725042093244416}r & \frac{63329157580689}{34900168372977664}r & \frac{305254689835095}{17450084186488832}r & \frac{-709134128594798907}{698003367459553280}r & r & \frac{780582495}{2052951080763392} \\ \frac{-4855508}{532534307449}r & \frac{1349019}{18363251981}r & \frac{-224107625}{532534307449}r & \frac{1166714500}{532534307449}r & \frac{-8646378300}{532534307449}r & \frac{-524865106367}{532534407449}r & 0 & r - \frac{17900}{31325547497} \end{bmatrix}$$

$$= r^8 - r^7 = 0$$

$r_1 = 1, r_2 = r_3 = r_4 = r_5 = r_6 = r_7 = r_8 = 0 < 1$
Hence, the block method (11) is zero stable and is of order $P = 10$ and hence by Henrici (1962) it is convergent.

NUMERICAL EXAMPLES AND RESULTS

The newly constructed continuous hybrid second derivative block backward differentiation formulae are applied in block forms for step number k = 7 to solve linear and nonlinear stiff systems of ordinary differential equations.

Problem 1

$$y_1' = -29998y_1 - 59994y_2$$
$$y_2' = 9999y_1 + 19997y_2$$

Exact $y_1(x) = \left(\frac{1}{9999}\right)(29997e^{-10000x}\text{-}19998e^{-x})y_1(0) = 1$

$y_2(x) =$ $-e^{-10000x} + e^{-x}y_2(0) = 0$

$$with\ h = 0.01$$

Problem 2

$$y_1' = -2y_1 + y_2 + 2sinx$$

$y_2' = 998y_1 - 999y_2 + 999(cosx - sinx)$ $\quad with\ h = 0.01$

Exact $y_{1(x)} = 2e^{-x} + sinxy_1(0) = 2$

$$y_2(x) = 2e^{-x} + cosxy_2(0) = 3$$

x	Exact Result (y_1)	Exact Result (y_2)	Computed Result (y_1)	Computed Result (y_2)	$Err(y_1)$	$Err(y_2)$
0.00	1.000000000000000	0.000000000000000	1.000000000000000	0.000000000000000	0.000000000000000	0.000000000000000
0.01	-2.14934213266561	1.046464043675030	-1.98009966749833	0.990049833749168	0.169242465167273	0.056414209925862
0.02	-1.95085001595262	0.977016280037103	-1.96039734661351	0.980198673306755	0.009547330660887	0.003182393269653
0.03	-1.94142998254088	0.970625221147549	-1.94089106709701	0.970445533548508	0.000538915443868	0.000179687599041
0.04	-1.92154877947126	0.960779454002632	-1.92157887830464	0.960789439152323	0.000030098833383	0.000009985149691
0.05	-1.90246084228349	0.951230135417767	-1.90245884900142	0.951229424500714	0.000001993282063	0.000000710917053
0.06	-1.88352924186737	0.941764637024608	-1.88352906716849	0.941764533584249	0.000000174698880	0.000000103440359
0.07	-1.86478790913450	0.932393953631854	-1.86478763981189	0.932393819905948	0.000000269322607	0.000000133725906
0.08	-1.84623294888172	0.923116474466264	-1.84623269277327	0.923116346386636	0.000000256108451	0.000000128079629
0.09	-1.82786261971605	0.913931309829498	-1.82786237054245	0.913931185271228	0.000000249173596	0.000000124558270
0.10	-1.80967507802436	0.904837538986969	-1.80967483607191	0.904837418035960	0.000000241952449	0.000000120951010

Computational Results of Problem 1 with Step Length 0.01

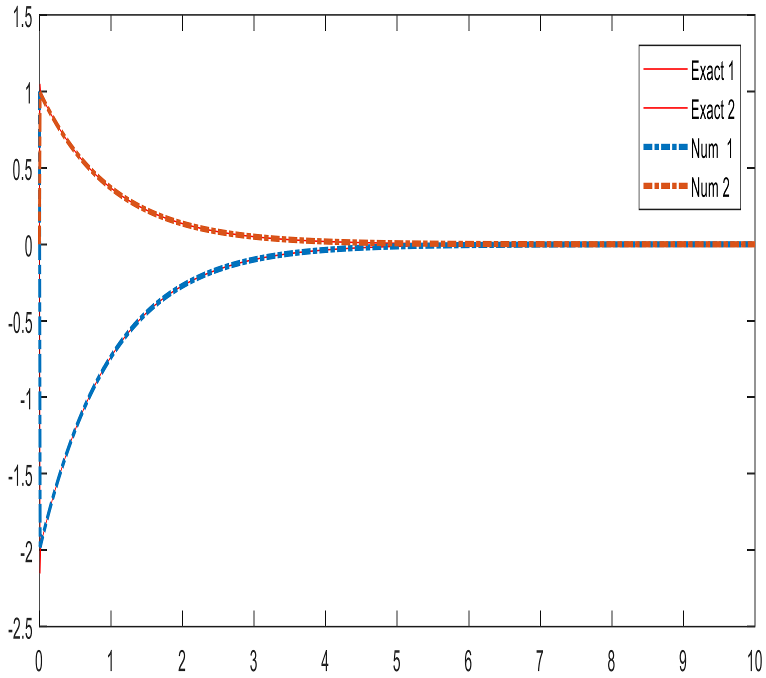

Figure 1: Solution Curve of Problem 1

x	Exact Result (y_1)	Exact Result (y_2)	Computed Result (y_1)	Computed Result (y_2)	$Err(\ y_1)$	$Err(\ y_2)$
0.00	2.000000000000000	3.000000000000000	2.000000000000000	3.000000000000000	0.000000000000000	0.000000000000000
0.01	1.990308688678580	2.981329394981550	1.990099500832500	2.980049667915000	0.000209187846079	0.001279727066554
0.02	1.980811325140600	2.961641405598490	1.980396013306840	2.960197353280080	0.000415311833760	0.001444052318409
0.03	1.971503794391690	2.942097384658440	1.970886567299510	2.940441100846000	0.000617227092185	0.001656283812437
0.04	1.962383221589890	2.922640811553030	1.961568212491280	2.920778984965620	0.000815009098614	0.001861826587410
0.05	1.953446694314420	2.903272374112960	1.952438018272100	2.901209109396390	0.001008676042320	0.002063264716568
0.06	1.944691322458600	2.883990104912360	1.943493073647940	2.881729607103700	0.001198248810665	0.002260497808667
0.07	1.936114235324660	2.864792191591850	1.934730487149420	2.862338640065170	0.001383748175237	0.002453551526680
0.08	1.927712581667710	2.845676844714690	1.926147386742440	2.843034399075890	0.001565194925270	0.002642445638808
0.09	1.919483529604040	2.826642303770020	1.917740919740460	2.823815103554450	0.001742609863574	0.002827200215575
0.10	1.911424266527850	2.807686836691370	1.909508252718740	2.804679001349940	0.001916013809109	0.003007835341427

Computational Results of Problem 2 with Step Length 0.01

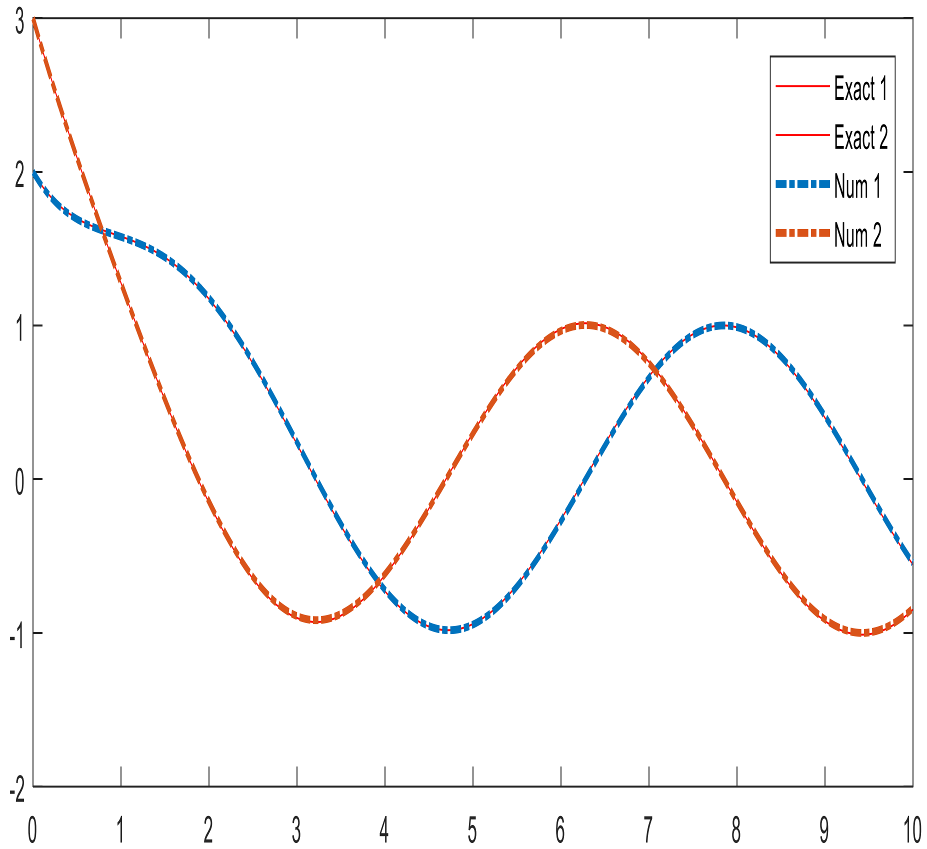

Figure 2: Solution Curve of Problem 2

CONCLUSION

In this paper, we proposed a seven-step extended block hybrid backward differentiation formulae for the solution of stiff ordinary differential equations, the developed scheme is of order ten which was applied to solve couples of stiff problems. Our analysis reveals that the newly constructed was found to be zero stable, consistent and converges. Two numerical examples were used to test the accuracy and efficiency of the developed method (see, Table 1 and 2). Our method proved to be promising as it produce the result of exact solution for the numerical example.

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