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3-Point Diagonally Implicit Super Class of Block Backward Differentiation Formula for Solving Stiff Initial Value Problems

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Abstract
A Diagonally Implicit 3-point block method for solving stiff initial value problems is developed. The method is derived by introducing a triangular matrix in the coefficient matrix of an existing fifth order implicit block method for solving first order stiff ordinary differential equations (3SBBDF). The order of the method is 3. The stability analysis indicates that the method is both zero and A-stable. The results obtained are compared with some existing algorithms and the performance of the scheme shows advantage in accuracy and computation time over some algorithms. The new method can serve as an alternative for solving some linear and non-linear stiff IVPs.

Keywords: Diagonally implicit block method, stiff, order, zero stability, block backward differentiation formula, A–Stability.

INTRODUCTION
As most explicit schemes became numerically unstable when applied for the solution of stiff initial value problems (IVPs), the need for the development of implicit numerical methods for the solution of stiff problems cannot be over emphasized. When integrating a differential equation numerically, the step size would be expected to be relatively small in a region where the solution curve displays much variation and to be relatively large where the solution curve straightens out to approach a line with slope nearly zero. For some problems, this is not the case. Sometimes, the step size may be forced down to an unacceptably small level in a region where the solution curve is very smooth. As observed by Ashi (2008), the phenomenon being exhibited is known as stiffness and it calls for implicit numerical schemes with A – stability properties.

According to Curtiss (1952), most of the available methods for solving stiff IVPs are based on Backward Differentiation Formula (BDF). There are many BDF based methods available in the literature which are either in form of sequential or block methods. Examples of the sequential methods are extended backward differential formula by Cash (1980),. modified extended backward differential formula by Cash (2000). Examples of block methods include block backward differentiation formula (BBDF) by Ibrahim *et al* (2007), 2 point diagonally implicit super class of backward differentiation formula by Musa *et al* (2016), diagonally implicit block backward differentiation formula for solving ODEs by Zawawi *et al* (2012), a new fifth order implicit block method for solving first order stiff ordinary differential

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equations by Musa *et tal* (2014), a new super class of block backward differentiation formula for stiff ordinary differential equations by Sueliam *et al (2014)*.. This paper considers the derivation of diagonally implicit form of the formula in Musa *et al* (2014), by introducing a lower triangular matrix in the formula. The use diagonally implicit method reduces the number of function evaluations in the computation; thereby reducing computation time.

DERIVATION

Consider the following numerical method developed by Musa *et al* (2014):

$$\sum_{j=0}^{5} \alpha_{j,i}\, y_{n+j-2} = h\beta_{k,i}(f_{n+k} - \rho f_{n+k-1}), \qquad k = i = 1,2,3 \tag{1}$$

where $\beta_{k,-1,i} = \rho\beta_{k,i}$, $\rho \in (-1, 1)$. Formula (1) is used for the integration of stiff IVPs and is known to be A–stable when ρ is chosen within the interval (-1, 1). The method is fully implicit and computes 3 solution values per step.

In this paper, a diagonally implicit form of (1) that also computes three solution values at a time is introduced as follows:

$$\sum_{j=0}^{2+k} \alpha_{j,i}\, y_{n+j-2} = h\beta_{k,i}(f_{n+k} - \rho f_{n+k-1}), \qquad k = 1,2,3 \tag{2}$$

$k = i = 1, k = i = 2$ and $k = i = 3$ represent the first, second and third points formulae respectively. Unlike the 3-point super class of BBDF (1), both the first and the second points of (2) have one less point than those in (1). This makes (2) to contain a lower triangular matrix in the coefficient matrix when the method is written in matrix form, thereby qualifying it to be diagonally implicit method.

The formula (2) is derived using Taylor's series expansion. A Linear operator L_i for the first, second and third point of (2) is defined by:

$$L_i[y(x_n), h]\colon \alpha_{0,i}y(x_n - 2h) + \alpha_{1,i}y(x_n - h) + \alpha_{2,i}y(x_n) + \alpha_{3,i}y(x_n + h) - h\beta_{k,i}(f(x_n + kh) - \rho f(x_n + (k-1)h)) = 0. \quad k = i = 1,2,3. \tag{3}$$

To derive the first point, set $k = i = 1$ in (3) to obtain:

$$L_1[y(x_n), h]\colon \alpha_{0,1}y(x_n - 2h) + \alpha_{1,1}y(x_n - h) + \alpha_{2,1}y(x_n) + \alpha_{3,1}y(x_n + h) - h\beta_{1,1}\big(f(x_n + h) - \rho f(x_n)\big) = 0, \tag{4}$$

Expanding (4) as Taylor's series about x_n, and collecting like terms, the following is obtained:

$$C_{0,1}y(x_n) + C_{1,1}hy'(x_n) + C_{2,1}h^2y''(x_n) + C_{3,1}h^3y'''(x_n) + \cdots \tag{5}$$

where

$$\left.\begin{aligned} C_{0,1} &= \alpha_{0,1} + \alpha_{1,1} + \alpha_{2,1} + \alpha_{3,1} = 0 \\ C_{1,1} &= -2\alpha_{0,1} - \alpha_{1,1} + \alpha_{3,1} + \beta_{1,1}(\rho - 1) = 0 \\ C_{2,1} &= 2\alpha_{0,1} + \tfrac{1}{2}\alpha_{1,1} + \tfrac{1}{2}\alpha_{3,1} - \beta_{1,1} = 0 \\ C_{3,1} &= -\tfrac{4}{3}\alpha_{0,1} - \tfrac{1}{6}\alpha_{1,1} + \tfrac{1}{6}\alpha_{3,1} - \tfrac{1}{2}\beta_{1,1} = 0 \end{aligned}\right\} \tag{6}$$

Solving (6) simultaneously, normalizing $\alpha_{3,1}$ and substituting the values of $\alpha_{j,1}$'s and $\beta_{j,1}$'s in (4) give the first point formula as:

$$y_{n+1} = -\frac{2+\rho}{-11+2\rho}y_{n-2} + \frac{3(3+2\rho)}{-11+2\rho}y_{n-1} - \frac{3(6+\rho)}{-11+2\rho}y_n + \rho h\frac{6}{-11+2\rho}f_n - \frac{6}{-11+2\rho}hf_{n+1} \tag{7}$$

Substituting $k = i = 2$ and $k = i = 3$ in (3) and following similar procedure as in the derivation of the first point, the second and the third points formulae are obtained. Therefore, the 3–point diagonally implicit super class of block backward differentiation formula (3DISBBDF) is obtained as:

$$\left.\begin{aligned} y_{n+1} &= -\frac{2+\rho}{-11+2\rho}y_{n-2} + \frac{3(3+2\rho)}{-11+2\rho}y_{n-1} - \frac{3(6+\rho)}{-11+2\rho}y_n + \rho h\frac{6}{-11+2\rho}f_n - \frac{6}{-11+2\rho}hf_{n+1} \\ y_{n+2} &= \frac{3+\rho}{-25+3\rho}y_{n-2} - \frac{2(8+3\rho)}{-25+3\rho}y_{n-1} + \frac{18(2+\rho)}{-25+3\rho}y_n - \frac{2(24+5\rho)}{-25+3\rho}y_{n+1} + \rho h\frac{12}{-25+3\rho}f_{n+1} - \frac{12}{-25+3\rho}hf_{n+2} \\ y_{n+3} &= -\frac{3(4+\rho)}{-137+12\rho}y_{n-2} + \frac{5(15+4\rho)}{-137+12\rho}y_{n-1} - \frac{20(10+3\rho)}{-137+12\rho}y_n + \frac{60(5+2\rho)}{-137+12\rho}y_{n+1} - \frac{5(60+13\rho)}{-137+12\rho}y_{n+2} \\ &\quad + \rho h\frac{60}{-137+12\rho}f_{n+2} - \frac{60}{-137+12\rho}hf_{n+3} \end{aligned}\right\} \tag{8}$$

Stability of the super class of BDF and BBDF methods with $\rho\epsilon(-1, 1)$ have been discussed in Vijitha-Kanaka (1985) and Musa (2013) respectively. For stability reasons, the method (8) is restricted to the value of ρ in the interval (0, 1) and any value of ρ within this interval can be used.

Substituting $\rho = \frac{9}{10}$ in equation (8), the 3DISBBDF is obtained as:

$$\left.\begin{aligned} y_{n+1} &= \frac{29}{92}y_{n-2} - \frac{36}{23}y_{n-1} + \frac{9}{4}y_n - \frac{27}{46}hf_n + \frac{15}{23}hf_{n+1} \\ y_{n+2} &= -\frac{39}{223}y_{n-2} + \frac{214}{223}y_{n-1} - \frac{522}{223}y_n + \frac{570}{223}y_{n+1} - \frac{108}{223}f_{n+1} + \frac{120}{223}f_{n+2} \\ y_{n+3} &= \frac{147}{1262}y_{n-2} - \frac{465}{631}y_{n-1} + \frac{1270}{631}y_n - \frac{2040}{631}y_{n+1} + \frac{3585}{1262}y_{n+2} - \frac{270}{631}f_{n+2} + \frac{300}{631}f_{n+3} \end{aligned}\right\} \tag{9}$$

The error constant of the method (9) is $C_4 = \begin{pmatrix} -\frac{15}{76} \\ 0 \\ 0 \end{pmatrix}$, indicating that the order of the method is 3.

Stability Analysis

The method (9) is written in matrix form as:

$$\begin{pmatrix} 1 & 0 & 0 \\ -\frac{570}{223} & 1 & 0 \\ \frac{2040}{631} & -\frac{3585}{1262} & 1 \end{pmatrix}\begin{pmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \end{pmatrix} = \begin{pmatrix} \frac{29}{92} & -\frac{36}{23} & \frac{9}{4} \\ -\frac{39}{223} & \frac{214}{223} & -\frac{522}{223} \\ \frac{147}{1262} & -\frac{465}{631} & \frac{1270}{631} \end{pmatrix}\begin{pmatrix} y_{n-2} \\ y_{n-1} \\ y_n \end{pmatrix} + h\begin{pmatrix} 0 & 0 & -\frac{27}{46} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} f_{n-2} \\ f_{n-1} \\ f_n \end{pmatrix}$$
$$+h\begin{pmatrix} \frac{15}{23} & 0 & 0 \\ -\frac{108}{223} & \frac{120}{223} & 0 \\ 0 & -\frac{270}{631} & \frac{300}{631} \end{pmatrix}\begin{pmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \end{pmatrix} \tag{10}$$

It can be noted that the coefficient matrix on the left hand side of (10) is a lower triangular matrix, hence qualifying the method as diagonally implicit.

Equation (10) can be rewritten in the following form:

$$A_0Y_m = A_1Y_{m-1} + h(B_0F_{m-1} + B_1F_m) \tag{11}$$

where $A_0 = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{570}{223} & 1 & 0 \\ \frac{2040}{631} & -\frac{3585}{1262} & 1 \end{pmatrix}$, $\quad A_1 = \begin{pmatrix} \frac{29}{92} & -\frac{36}{23} & \frac{9}{4} \\ -\frac{39}{223} & \frac{214}{223} & -\frac{522}{223} \\ \frac{147}{1262} & -\frac{465}{631} & \frac{1270}{631} \end{pmatrix}$,

$B_0 = \begin{pmatrix} 0 & 0 & -\frac{27}{46} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\quad B_1 = \begin{pmatrix} \frac{15}{23} & 0 & 0 \\ -\frac{108}{223} & \frac{120}{223} & 0 \\ 0 & -\frac{270}{631} & \frac{300}{631} \end{pmatrix}$,

Substituting the scalar test equation

$$y' = \lambda y \tag{12}$$

($\lambda<0$, λ complex) into (11) and using $\lambda h=\bar{h}$ gives

$$A_0 Y_m = A_1 Y_{m-1} + \bar{h}(B_0 Y_{m-1} + B_1 Y_m) \tag{13}$$

The stability polynomial of (9) is obtained by evaluating

$$\mathrm{Det}\left[(A_0 - \bar{h}B_1)t - (A_1 + \bar{h}B_0)\right] = 0 \tag{14}$$

to obtain:

$$R(t,\bar{h}) = \frac{81609}{221312}t + \frac{3267}{442624}\bar{h} + \frac{10323}{17024}t\bar{h} - \frac{1224519}{885248}t^2 - \frac{175437}{221312}\bar{h}t^2 - \frac{150795}{221312}\bar{h}^2t^2 + \frac{12123}{13832}\bar{h}^3t^3 - \frac{270}{1729}\bar{h}^3t^3\bar{h}^2t + \frac{29079}{221312}\bar{h}^2t + \frac{3645}{55328}\bar{h}^3t^2 + t^3 + \frac{12835}{885248} - \frac{90063}{55328}\bar{h}t^3 = 0 \tag{15}$$

Substituting $\bar{h} = 0$, the first characteristic polynomial is obtained as:

$$\frac{66745}{3236399} + \frac{2199921}{3236399}t - \frac{5503065}{3236399}t^2 + t^3 = 0 \tag{16}$$

Solving equation (16) for t gives the following roots:
t=1, t= 0.7286692660, t = 0.028302593.
The values obtained indicate that the method is zero stable.
The stability region of the method (9) as obtained using Maple software is:

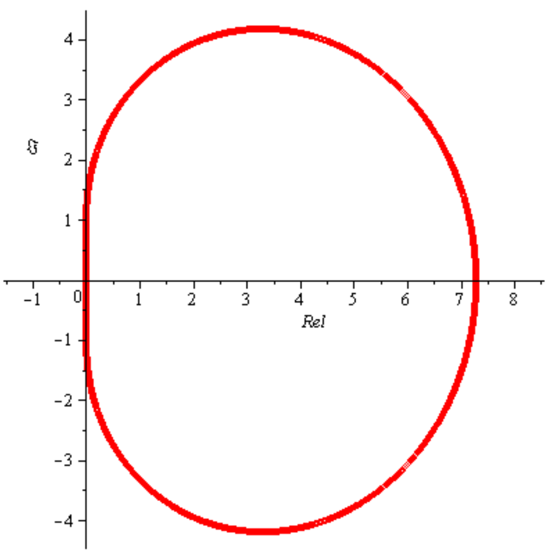

Figure 1: Stability Region of 3DISBBDF for $\rho = \frac{9}{10}$

The stability region is the region outside the circular shape, and thus covers the entire negative half plane. This indicates A– stability of the method, which is an essential property desired for solving stiff IVPs.

Implementation of the method

Newton's iteration is employed to implement the method. Let y_i and $y(x_i)$ be respectively the approximate and exact solution of the stiff IVP:

$$y' = f(x,y), \qquad y(x_0) = y_0, \qquad x \in (a,b) \tag{17}$$

Define the error as:

$$(error_i)_t = |(y_i)_t - (y(x_i))_t|. \tag{18}$$

and the maximum error as:

$$MAXE = \underbrace{max}_{1\le i\le T}\ (\underbrace{max(error_i)_t}_{1\le i\le N}). \tag{18}$$

where T is the total number of steps and N is the number of equations (see Ibrahim et al (2007)).

Let

$$\left.\begin{aligned} F_1 &= y_{n+1} + \frac{27}{46}hf_n - \frac{15}{23}hf_{n+1} - \varepsilon_1 \\ F_2 &= y_{n+2} - \frac{570}{223}y_{n+1} + \frac{108}{223}hf_{n+1} - \frac{120}{223}hf_{n+2} - \varepsilon_2 \\ F_3 &= y_{n+3} + \frac{2040}{631}y_{n+1} - \frac{3585}{1262}y_{n+2} + \frac{270}{631}hf_{n+2} - \frac{300}{631}hf_{n+3} - \varepsilon_3 \end{aligned}\right\} \tag{19}$$

where

$$\left.\begin{aligned} \varepsilon_1 &= \frac{29}{92}y_{n-2} - \frac{36}{23}y_{n-1} + \frac{9}{4}y_n \\ \varepsilon_2 &= -\frac{39}{223}y_{n-2} + \frac{214}{223}y_{n-1} - \frac{522}{223}y_n \\ \varepsilon_3 &= \frac{147}{1262}y_{n-2} - \frac{465}{631}y_{n-1} + \frac{1270}{631}y_n \end{aligned}\right\} \tag{20}$$

are the back values.

Let $y_{n+1}^{(i+1)}$ denote the (i+1)th iteration and

$$e_{n+j}^{(i+1)} = y_{n+j}^{(i+1)} - y_{n+j}^{(i)}, \qquad j=1,2,3. \tag{21}$$

Applying Newton's iteration, we have

$$e_{n+j}^{(i+1)} = -[F_j'(y_{n+j}^{(i)})]^{-1}[F_j(y_{n+j}^{(i)})], \quad j=1,2,3. \tag{22}$$

Equation (22) can be written in the form:

$$\left[F_j'\left(y_{n+j}^{(i)}\right)\right] e_{n+j}^{(i+1)} = -[F_j(y_{n+j}^{(i)})], \quad j=1,2,3. \tag{23}$$

and in matrix form, equation (23) is equivalent to:

$$\begin{bmatrix} 1-\frac{15}{23}h\frac{\delta f_{n+1}}{\delta y_{n+1}} & 0 & 0 \\ -\frac{570}{223}+\frac{108}{223}h\frac{\delta f_{n+1}}{\delta y_{n+1}} & 1-\frac{120}{223}h\frac{\delta f_{n+2}}{\delta y_{n+2}} & 0 \\ \frac{2040}{631} & -\frac{3585}{1262}+\frac{270}{631}h\frac{\delta f_{n+2}}{\delta y_{n+3}} & 1-\frac{300}{631}h\frac{\delta f_{n+3}}{\delta y_{n+3}} \end{bmatrix}\begin{bmatrix} e_{n+1}^{(i+1)} \\ e_{n+2}^{(i+1)} \\ e_{n+3}^{(i+1)} \end{bmatrix} =$$

$$\begin{bmatrix} -1 & 0 & 0 \\ \frac{570}{223} & -1 & 0 \\ \frac{-2040}{631} & \frac{3585}{1262} & -1 \end{bmatrix}\begin{bmatrix} y_{n+1}^{(i)} \\ y_{n+2}^{(i)} \\ y_{n+3}^{(i)} \end{bmatrix} + h\begin{bmatrix} 0 & 0 & -\frac{27}{46} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} f_{n-2}^{(i)} \\ f_{n-1}^{(i)} \\ f_{n}^{(i)} \end{bmatrix} + h\begin{bmatrix} \frac{15}{23} & 0 & 0 \\ -\frac{108}{223} & \frac{120}{223} & 0 \\ 0 & -\frac{270}{631} & \frac{300}{631} \end{bmatrix}\begin{bmatrix} f_{n+1}^{(i)} \\ f_{n+2}^{(i)} \\ f_{n+3}^{(i)} \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix} \tag{24}$$

A computer code is designed to implement (24).

Test Problems

The following stiff IVPs are considered for testing the performance of the method.

Problem 1.

$y_1' = 198y_1 + 199y_2,$ $\quad y_1(0) = 1$ $\quad 0 \le x \le 10$

$y_2' = -398y_1 - 399y_2.$ $\quad y_2(0) = -1$

Exact solution:

$y_1(x) = e^{-x}$

$y_2(x) = -e^{-x}.$

Eigen values: -1 and -200

Source: Ibrahim *et al.* (2007).

Problem 2.

$y_1' = 32y_1 + 66y_2 + \frac{2}{3}x + \frac{2}{3}$ $\quad y_1(0) = \frac{1}{3}$ $\quad 0 \le x \le 1.$

$y_2' = -66y_1 - 133y_2 - \frac{1}{3}x - \frac{1}{3}$ $\quad y_2(0) = \frac{1}{3}$

Exact solution

$y_1(x) = \frac{2}{3}x + \frac{2}{3}e^{-x} - \frac{1}{3}e^{-100x}$

$y_2(x) = -\frac{1}{3}x - \frac{1}{3}e^{-x} + \frac{2}{3}e^{-100x}$

Eigen values -1 and -100.

Source: Musa *et al.* (2016)

Problem 3.

$y' = -10xy.$ $\quad y(0) = 1.$ $\quad 0 \le x \le 10$

Exact solution:

$y(x) = e^{-5x^2}$

Source : Musa *et al.* (2014).

Numerical Results and Discussion

The test problems presented are solved with the proposed method, the 3 point block backward differentiation formula and the 3 point super class of block backward differentiation formula. for comparison purposes. The results are presented in Table 1–3. The graph indicating the performance of each method is also plotted and presented in Figure 1–3. For fairness in reporting the computation time, all problems are solved on the same computer machine with the specification 1.80 GHz, 4.00BG RAM and 62 bit operating system.

The following notations are used in the tables.

3BBDF: 3 point block backward differentiation formula

3SBBDF: 3 point super class of block backward differentiation formula
3DISBBDF: 3 point diagonally implicit super class of block backward differentiation formula
h: Step size
NS: Total number of steps
T: Time in s.

Table 1: Numerical results for Problem 1.

h	Method	NS	MAXE	T(s)
10^{-2}	3BBDF	333	1.07308e-002	1.23900e-002
	3SBBDF	333	1.94447e-004	1.20394e-002
	DI3SBBDF	333	4.72745e-004	1.10813e-002
10^{-3}	3BBDF	3,333	1.10060e-003	1.89200e-002
	3SBBDF	3,333	2.07993e-006	1.19193e-001
	DI3SBBDF	3,333	5.88650e-006	3.69500e-002
10^{-4}	3BBDF	33,333	1.10333e-004	7.70200e-002
	3SBBDF	33,333	2.09995e-008	1.19296e+000
	DI3SBBDF	33,333	6.12465e-008	2.74800e-001
10^{-5}	3BBDF	333,333	1.10361e-005	6.74400e-001
	3SBBDF	333,333	2.10257e-010	1.19173e+001
	DI3SBBDF	333,333	6.16220e-010	2.54300e+000
10^{-6}	33BBDF	3,333,333	1.10363e-006	6.63000e+000
	3SBBDF	3,333,333	1.41029e-011	1.19110e+002
	DI3SBBDF	3,333,333	7.34081e-010	2.54500e+001

Table 2: Numerical results for Problem 2.

h	Method	NS	MAXE	T(s)
10^{-2}	3BBDF	333	1.12578e-002	9.10500e-001
	3SBBDF	333	1.21523e-002	2.80463e-002
	DI3SBBDF	333	1.21469e-002	2.13500e-002
10^{-3}	3BBDF	3,333	4.97329e-002	3.29800e-002
	3SBBDF	3,333	8.66386e-003	2.79386e-001
	DI3SBBDF	3,333	1.26795e-003	2.96400e-002
10^{-4}	3BBDF	33,333	7.15289e-004	1.96300e-002
	3SBBDF	33,333	1.29624e-004	2.78476e-000
	DI3SBBDF	33,333	3.15144e-004	6.98500e-001
10^{-5}	3BBDF	333,333	7.33633e-004	1.11330e-001
	3SBBDF	333,333	1.38655e-007	2.78828e+001
	DI3SBBDF	333,333	3.92413e-006	6.84500e+000
10^{-6}	3BBDF	3,333,333	7.35458e-005	1.04700e+000
	3SBBDF	3,333,333	1.39989e-008	2.78526e+002
	DI3SBBDF	3,333,333	4.08289e-008	6.83600e+001

Table 3: Numerical results for Problem 3.

h	Method	NS	MAXE	T(s)
10^{-2}	3BBDF	333	3.56692e-002	1.33900e-002
	3SBBDF	333	2.01743e-003	1.28000e-002
	3DISBBDF	333	4.5886e-003	8.50400e-003
10^{-3}	3BBDF	3,333	4.28514e-003	1.10990e-001
	3SBBDF	3,333	2.10059e-005	3.47700e-002
	3DISBBDF	3,333	6.06383e-005	1.35600e-002
10^{-4}	3BBDF	33,333	4.35640e-004	2.81270e-001
	3SBBDF	33,333	2.10289e-007	2.46500e-001
	3DISBBDF	33,333	6.16348e-007	2.67000e-002
10^{-5}	3BBDF	333,333	4.36353e-005	2.91110e-001
	3SBBDF	333,333	2.10294e-009	2.29700e+000
	3DISBBDF	333,333	6.16613e-009	1.68200e-001
10^{-6}	3BBDF	3,333,333	4.36425e-006	3.03120e-001
	3SBBDF	3,333,333	2.10281e-011	2.27900e+001
	3DISBBDF	3,333,333	3.45228e-010	1.58700e+000

In order to give a visual impact of the performance of the methods, the graphs of $Log_{10}MAXE$ against h for the problems tested are plotted; giving a scaled maximum error for the different problems tested.

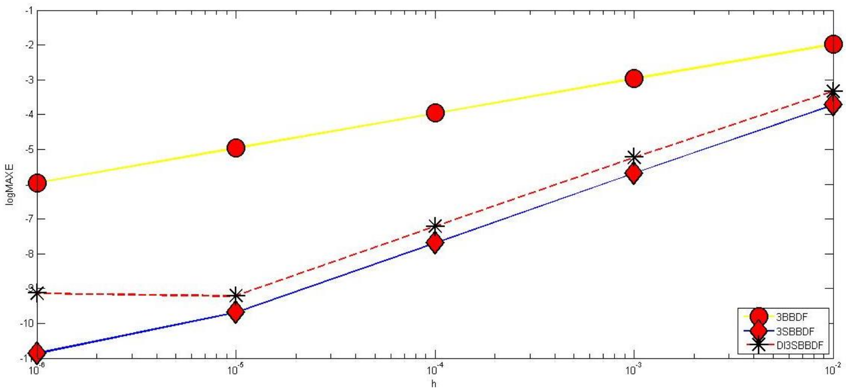

Figure 2: Graph of $Log_{10}MAXE$ against 'h' for Problem 1.

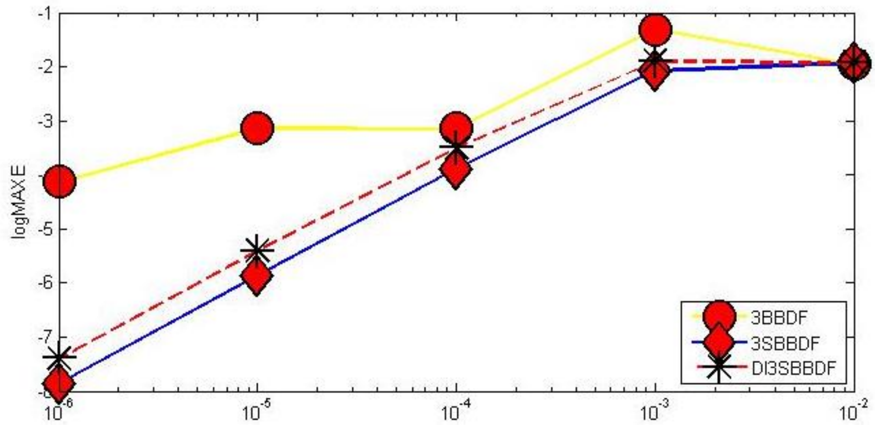

Figure 3: Graph of $Log_{10}MAXE$ against 'h' for Problem 2.

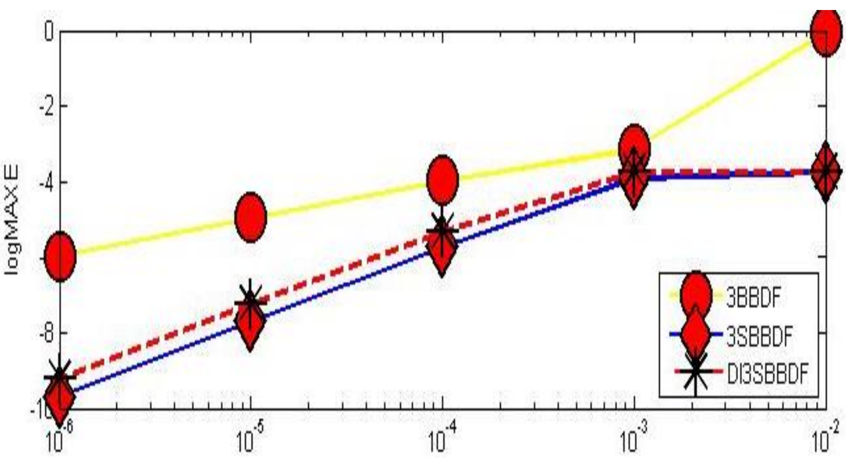

Figure 4: Graph of $Log_{10}MAXE$ against 'h' for Problem 3.

From Tables 1− 3, it can be seen that the 3DISBBDF method outperformed the 3BBDF method and competes with the 3SBBDF method in terms of accuracy. Convergence is evident in the developed method by the decrease in error as the step length h tends to zero. Similarly, the solution at any fixed point improves as the step length is reduced. Thus, the computed solution tends to the exact solution as the step length tends to zero. The computation time of the new method is better than that of 3SBBDF. However, the computation time in BBDF method is better than the time in the new method.

The graphs in Figure 2 – 4 also show that the scaled error for the 3DISBBDF is smaller when compared with that in 3BBDF method and are competing with those of 3SBBDF. This is why the graph of the 3DISBBDF is below that of the BBDF and close to that of 3SBBDF.

CONCLUSION

A new numerical method called Diagonally Implicit 3-point Super Class of Block Backward Differentiation Formula (DI3SBBDF) for solving stiff IVPs is developed. It computes three solution values per step. The developed method is both zero stable and A-stable. The method was tested by solving some stiff initial value problems. Results obtained indicate that the new method outperformed 3BBDF and competes favourably with the existing 3SBBDF method in terms of accuracy. For computation time, the new method outperformed the 3SBBDF, and has no advantage over 3BBDF.

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