ISSN (Print): 2476-8316
ISSN (Online): 2635-3490
Dutse Journal of Pure and Applied Sciences (DUJOPAS), *Vol. 5 No. 1a June 2019*

Gravitomagnetic Coupling in a Weak Field Approximation

Sarki M. U. *[1,3], Ewa I. I. [1], Mustapha I. M [1], Yusuf M. A. [2] Isah S.H [4]

[1]Department of Physics,
Nasarawa State University,
PMB 1022, Keffi, Nasarawa State.
musarki@nsuk.edu.ng

[2]Department of Physics,
Abubakar Tafawa Balewa University,
PMB 0248 Bauchi State.

[3]Federal University of Agriculture,
PMB 2373, Makurdi, Benue State.

[4]Department of Mathematics,
Nasarawa State University,
PMB 1022, Keffi, Nasarawa State.

Abstract

According to general relativity, a moving or rotating matter should produce contribution to the gravitational field which is in analogy to the magnetic field of a moving charge or magnetic dipole; we can thus express the general relativity in a Maxwellian structure with the existence of single operator for both fields. In this research work, we studied the coupling of Maxwell's dynamical theory of electromagnetism in a non-static gravitational field by applying an obtained gravitational scalar potential to the Riemann's laplacian operator. In the limit of weak gravitation, we obtained the dynamical gravitomagnetic scalar potential across the fields. While in the boundary conditions the results are generalized gravitomagnetic potential. our results will thus widen the scope into study of gravitomagnetic coupling.

Keywords: Gravitomagnetism, Coupling, Scalar Potential, Vector potential, Riemann's Laplacian Operator

INTRODUCTION

Einstein's geometrical field equation's EGFE are sets of equations in General Relativity GR which describe the fundamental interaction of gravitation as a manifestation of curved space and time. EFGE was first published by Einstein in 1915, General relativity has been a widely accepted theory of gravitation. linearization in the weak and slow motion approximation has in previous literatures unveil a structural solution of EGFE to a sets of second order partial differential equation comparable to Maxwell's equations of electromagnetism, as a consequence of gravitomagnetic field effect induced by off-diagonal component $g_{\mu\nu}$of the space time metric tensor (Lorio & Corda, 2011).

**Author for Correspondence*

According to general relativity the rotation of the earth produces a gravitomagnetic field, and its influence on other planets was first investigated by de Sitter and later generalized by Lense and Thirring. Several studies have been made to find the formal analogy between Maxwell's equations and the gravitation theory, Heaviside predicted the gravitomagnetic field (Youssef, 2017). Holzumuler and Tisser and postulated that the gravitational influence of the sun has an additional magnetic component. This formal analogy unveil some well-known experimental effects such as the lense-thirring precession, electromagnetic delay, frame dragging as well as change in phase of electromagnetic waves, which is a modification of the Newtonian idea of gravity to Einstein Relativity Theory.

In this article we apply the scalar potential obtained by Chifu(2009) to Maxwell's dynamical equations in the limit of weak gravitation to study gravitomagnetic scalar potential and vector potential.

Theoretical Background

The covariant metric tensor exterior to a non static varying distribution of mass within the region of spherical geometry is given by (Chifu, 2009).

$$g_{00} = 1 + \frac{2f(t,r)}{c^2} \tag{1}$$

$$g_{11} = -\left(1 + \frac{2f(t,r)}{c^2}\right)^{-1} \tag{2}$$

$$g_{22} = -r^2 \tag{3}$$

$$g_{33} = -r^2 \sin^2\theta \tag{4}$$

$$g_{\mu\nu} = 0; \tag{5}$$

Metric tensors are the starting point for all geometrical theories, this metric tensor should thus satisfy the Schwarzschild's metric, satisfy equivalence principle of Physics and should naturally reduce to second order differential equation (Howusu,2010a; Howusu, 2010b).

It could thus be remark that in the weak field limit c^0 the equation reduces to the well-known d'Alembertian operator.

$$\nabla^2 f(t,r) + \frac{1}{c^2}\left[1 + \frac{1}{c^2} f(t,r)\right]^{-2} \frac{\partial^2 f(t,r)}{\partial t^2} - \frac{4}{c^2}\left[1 + \frac{1}{c^2} f(t,r)\right]^{-3} \left(\frac{\partial f(t,r)}{\partial t}\right) = 0 \tag{6}$$

To the order of c^2, the geometrical wave equation in (6) reduces, in the limit of weak gravitational field to

$$\nabla^2 f(t,r) + \frac{1}{c^2} \partial_t^2 f(t,r) = \dot{\square} = 0 \tag{7}$$

where $\dot{\square}$ is d'Alembertian operator

This obtained d'Alembertian operator clearly unveils a structural operator comparable to Maxwell's electromagnetic field equation in equation (11)

By series linearization, the gravitational scalar potential was obtained as (Chifu, 2005).

$$f(t,r) \approx -\frac{k}{r} exp\, i\, \omega \left(t - \frac{r}{c}\right) \tag{8}$$

Methodology
Magnetic fields are generated by steady time –independent currents and satisfy the gauss's law for magnetic field (William& John, 2012; Young, 2008).

$$\nabla \cdot \mathbf{B} = 0 \tag{9}$$

Where **B** is the magnetic field defined by(Jackson, 1999; Steward, 2013; Nyambana, 2005).

$$\mathbf{B} = \nabla \times A(r,t) \tag{10}$$

Since the divergence of a curl is zero, and by computation from Amperes law it could be shown that equation (10) reduces to (Gupta, 2010; Nyambana, 2015).

$$\dot{\Box}_H \mathbf{A}(r,t) = \mu_0 \mathbf{J} \tag{11}$$

The Riemannian geometry form to equation (11) is given as(Gupta, 2010; Howusu, 2010a).

$$\dot{\Box}_H \mathbf{A}(r,t) = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\left(1+\frac{2}{c^2}f(r,t)\right)\frac{\partial \mathbf{A}(r,t)}{\partial r}\right] - \frac{1}{c^2}\frac{\partial}{\partial t}\left(1+\frac{2}{c^2}f(r,t)\right)\frac{\partial \mathbf{A}(r,t)}{\partial t} = \mu_0 \mathbf{J} \tag{11}$$

Where **J** is the current density, $\mu_0$is permeability of free space and $\mathbf{A}(r,t)$ magnetic vector potential.
Equation (11) is a Poisson-like equation and using a series linear approximation to obtain the

non-static vector potential, the non-static gravitational scalar potential in equation (11) could be transform to (Lumbi *et al.*, 2018).

$$f(t,r) = \frac{k}{r}\left[1+\frac{\omega^4}{8}\left(t^4 - \frac{4t^3 r}{c} + \frac{6t^2 r^2}{c^2}\right)\right] \tag{12}$$

Therefore differentiating equation (12) with respect to radial distance gives

$$f^1 = \frac{-k}{r^2}\left[1+\frac{\omega^4}{8}\left(t^4 - \frac{4t^3 r}{c} + \frac{6t^2 r^2}{c^2}\right)\right] + \frac{k\omega^4}{8r}\left[\left(\frac{12t^2 r}{c^2} - \frac{4t^3}{c}\right)\right] \tag{13}$$

Substituting (12) and (13) into (11) while taking the weak field limit of c^0 it reduces to the pure Poisson's equation given by

$$\frac{2}{r}A'(r,t) + A''(r,t) = \mu_o J \tag{14}$$

while in the weak limit of c^2 it reduces to

$$\left[\frac{-2k}{c^2 r^2} - \frac{k\omega^4 t^4}{4c^2 r}\right]A(r,t) + \left[\frac{2}{r} + \frac{4k}{c^2 r^2} - \frac{k\omega^4 t^4}{4c^2 r^2}\right]A'(r,t) + \left[1 + \frac{2k}{c^2 r} - \frac{k\omega^4 t^4}{4c^2 r}\right]A''(r,t) - \frac{1}{c^2}A_{(r,t)} = \mu_0 J \tag{15}$$

We seek a series approximation and apply a boundary conditions to the exterior field in the form as

$$A^+(r,t) = \frac{A_1}{r} + \frac{A_2}{r^2} + \ldots \tag{16}$$

Differentiating (16) with respect to r gives

$$A^{+\prime}(r,t) = -\frac{A_1}{r^2} - \frac{2A_2}{r^3} + \ldots \tag{17}$$

$$A^{+\prime\prime}(r,t) = \frac{2A_1}{r^3} - \frac{6A_2}{r^2} + \ldots \tag{18}$$

Substituting equation (16),(17) and (18) in (15) gives

$$\left[\frac{-2k}{c^2r^2}-\frac{k\omega^4t^4}{4c^2r}\right]\left(\frac{A_1}{r}+\frac{A_2}{r^2}\right)+\left[\frac{2}{r}+\frac{4k}{c^2r^2}-\frac{k\omega^4t^4}{4c^2r^2}\right]\left(-\frac{A_1}{r^2}-\frac{2A_2}{r^3}\right)$$

$$+\left[1+\frac{2k}{c^2r}-\frac{k\omega^4t^4}{4c^2r}\right]\left(\frac{2A_1}{r^3}-\frac{6A_2}{r^2}\right)-\frac{1}{c^2}\left(\frac{A_1}{r}+\frac{A_2}{r^2}\right)=\mu_0 J \quad (19)$$

Comparing the coefficient of $\frac{1}{r},\frac{1}{r^2},\frac{1}{r^3}$ with the condition of linear dependence of r on A

A_1= constant

We obtain an expression for A_1interms of A_2, A_2 interms of A_1 given as

$$A_I=\frac{\omega^4t^4A_2}{8}$$

$$A_2=\frac{8A_1}{\omega^4t^4} \quad (20)$$

$$A^{+}(r,t)=\frac{A_1}{r}\left(1-\frac{8}{\omega^4t^4r}\right)+\ldots \quad (21)$$

For a weak field approximation, the gauge approximation tends to be the linear perturbation of the interior field, given by (Bezerra *et al.*, 2005).

$$A^{-}(r,t)=A^{\chi}+A^{\zeta} \quad (22)$$

$$A^{\zeta}=B_0= \text{constant} \quad (23)$$

$$A^{-}=D_2r^2+D_4r^2+\ldots. \quad (24)$$

Differentiating (24) gives

$$A^{-\prime}=2D_2r+4D_4r^3+\ldots. \quad (25)$$

$$A^{-\prime\prime}=2D_2+12D_4r^2+\ldots. \quad (26)$$

Substituting (24) - (26) in equation (15)

$$\left[\frac{-2k}{c^2r^2}-\frac{k\omega^4t^4}{4c^2r}\right](D_2r^2+D_4r^2)+\left[\frac{2}{r}+\frac{4k}{c^2r^2}-\frac{k\omega^4t^4}{4c^2r^2}\right](2D_2r+4D_4r^3)+$$

$$\left[1+\frac{2k}{c^2r}-\frac{k\omega^4t^4}{4c^2r}\right](2D_2+12D_4r^2)-\frac{1}{c^2}(D_2r^2+D_4r^2)=\mu_0 J \quad (27)$$

Comparing the coefficients of r^0, while applying the physical relationship of current density J between field and density in a magnetic field system of radius R, and after a lengthy substitution of the perturbed terms we obtain

$$D_2=\frac{3\mu_0\Phi_B}{4\pi R^3} \quad (28)$$

$$D_4=0 \quad (29)$$

Adopting the gauge transformation

$$A\rightarrow A^{+}=A^{-} \quad (30)$$

$$A^{+}=A^{-} \quad (31)$$

It implies that equation (31) will be

$$B_0+D_2r^2=\frac{A_1}{r}+\frac{A_2}{r^2} \quad (32)$$

Magnetic scalar potential is a normal continuous function, the conditions of continuity of functions holds

$$\left(\frac{\partial A^{+}}{\partial r}\right)_{r=R} = \left(\frac{\partial A^{-}}{\partial r}\right)_{r=R} \tag{33}$$

$$2D_2R = -\frac{A_1}{R^2} - \frac{2A_2}{R^3} \tag{34}$$

Equation (34) could thus be written as

$$2\left(\frac{3\mu_0\Phi_B}{4\pi R^3}\right)R = \frac{A_1}{R^2} + \left(\frac{16A_I}{\omega^4t^4R^3}\right) \tag{35}$$

Equation (35) gives an expression for A_1 and thus A_2, using equation (20)
Substituting (31) into (21) , we obtain

$$A^{+} = \frac{3\mu_0\Phi_B}{2\pi r}\left[\frac{1}{2r} - \frac{4}{\omega^4t^4} - 1\right]\left[\frac{16}{\omega^4t^4R} - 1\right]^{-1} \tag{36}$$

From equation (32) we obtain the value for $B_o$by substituting A_1, A_2, D_2
Substituting the values of B_o, A_1, A_2, D_2 into equation (32) the interior field reduces to

$$A^{-} = \mu_0 J \tag{37}$$

Equation (36) and (37) are the gravitomagnetic vector potentials exterior and interior to the system.

Thus for an exterior field the non-relativistic gravitomagnetic potentials is

$$V_{g\Phi} = qA^{+} \tag{38}$$

$$V_{g\Phi} = \frac{3q\mu_0\Phi_B}{2\pi r}\left[\frac{1}{2r} - \frac{4}{\omega^4t^4} - 1\right]\left[\frac{16}{\omega^4t^4R}\right]^{-1} \tag{39}$$

Equation (38) is the total gravitomagnetic potential energy due to the exterior field. $\Phi_B$is the gravitomagnetic scalar potential, we can similarly express equation (39) in terms of the gravitomagnetic scalar potential.

Our obtained potential is indeed a profound discovery, as recent research have found potentials useful in molecular physics and quantum chemistry to study the molecular vibration-rotation of energy spectrum of linear and non-linear systems.

Our potential could be apply to time dependent Schrödinger equations, to study the energy spectrum of quantum field interactions in the presence of gravitomagetic field.

$$E\psi_{(r,t)} = \frac{\hbar}{2m}\nabla^2_{(r,t)} + V_{(r,t)}\psi_{(r,t)}$$

Our obtained potentials has a close analogy to Hartmann Potential in quantum mechanics for study of diatomic molecules, given by$V_{(r,\theta)} = \frac{Ze^2}{r} + \frac{B}{r^2\sin^2\theta} + \frac{\gamma\cos}{r^2\sin^2\theta}$

Woods saxon potential in nuclear physics for the distribution of nuclear densities, given by

$$V_{(r)} = -\frac{V_0}{1 + e^{2\alpha r}}$$

CONCLUSION

Thus, in this article we have seen the interaction between non-static Einstein geometrical field equation and Maxwell's dynamical field equation, thus the coupling effect is well established and our proposed study satisfies poisons equations given by (11), thus equivalence principle of Physics is well established.

Despite the discrete theoretical difference between Einstein's geometrical theory of gravitation and Maxwell's dynamical theory of electromagnetism i.e. the former guided by geometrical quantities and the later by dynamical quantities, the coupling effects will help us see their similarity and the possible unification of all fields in nature, this should be the achievable goal to all Physicists.

Acknowledgement

We thank the reviewers for their positive contribution which improve the manuscript. In addition M.U. Sarki acknowledge the continuous academic impact of Prof. E.N. Chifu.

REFERENCES

Bezerra, V.B., Barros, A and Romero, C (2005). On some Aspects of Gravitomagnetism in Scalar-Tensor Theories of Gravity, *Brazilian Journal of Physics*, 35:4b:1057-1061.

Chifu, E.N (2009). Solutions of Einstein's Geometrical Gravitational Field Equations Exterior to Astrophysically Real or Hypothetical Time varying Distributions of Mass within Regions of Spherical Geometry, *Progress in Physics*,3:45-48.

Gupta, B.D (2010). Mathematical Physics, Vikas Publishing HousePVT LTD, New Delhi.

Howusu, S.X.K. (2009).The Metric Tensors For Gravitational Filed and The Mathematical Princicples of Theoretical Physics,Jos University Press ltd, Plateau State Nigeria.

Howusu, S.X.K (2010a). Solution of Einstein's Geometrical Gravitational Field Equations, Jos University Press ltd, Plateau State Nigeria.

Howusu, S.X.K (2010b). Complete Dynamical Theories of Physics, Jos University Press limited, Plateau State, Nigeria.

Hugh D. Young & Roger A. Freedman (2008). University Physics,Pearson AddisonWesley, New York.

Jackson, J.D. (1999). Classical Electrodynamics, John Wiley and Sons, New York.

Lumbi, L.W., Uchenna, E.T., Ewa, I.I., Loko, A.Z., Ndkilar, Chifu E. (2018). An extended Study of Motion in the Vicinity of Non Static Spherical Gravitational Potential Fields, *Dutse Journal of Pure and Applied Science (*DUJOPAS), 4(2): 188-195.

Nyambana, G.G. (2015). Fundamental Physical Basis For Maxwell-Heaviside Gravitomagnetism, *Journal of Modern Physics*, 6; 1207-1219.

Oyewumio, K.J., Titiloye, E.O., Alabi, O.A & Falaye, B.J (2016). Bound States of Pseudo-Harmonic Oscillator in the Presence of Magnetic Field, Journal of the Nigerian MathematicalSociety, 35;460-467.

Stewart, A.M. (2003). Vector Potential of The Coulomb Gauge, *European Journal of Physic,* 24;519-524.

Tajmar, M. (2001). Coupling of Electromagnetism and Gravitation in the Weak Field Approximation, *Journal of Theoretics*,1-8.

William, H. Hayt, Jr. & John A. Back. (2012). Engineering Electromagnetics, McGraw Hill International Edition, Singapore.

Youssef, M. H. A. (2017). Short-Period and Long-Period Effects of Weak Gravitational Waves, *International Journal of Astronomy and Astrophysics*, 7; 230-237.