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Counting Distinct Fuzzy Subgroups of Dihedral Groups of Order $2p^nq$ where p and q are distinct primes

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Abstract
This paper gives an explicit formula for counting the number of distinct fuzzy subgroups of dihedral groups of order $2p^nq$, where p and q are distinct primes with respect to a new equivalence relation $\approx$..

Keywords: Dihedral group; Fuzzy subgroups; Equivalence relation

1. Introduction
Classification of elements of a set is basically to divide the set into distinct subsets usually called classes. This notion leads to the concept of equivalence relation on the set. Classifying fuzzy subgroups of finite groups is a fundamental problem of fuzzy group theory, see Murali and Makamba (2001 and 2003) . Without any equivalence relation on fuzzy subgroups of group G, the number of fuzzy subgroups is infinite, even for trivial group.

Research had focused on classification of fuzzy subgroups of dihedral groups D_{2p^nq} with respect to the equivalence relation $\sqcup$, thus, an explicit formula for counting the number of distinct fuzzy subgroups of D_{2p^nq} had been derived with respect to $\sqcup$, see Tarnuaceanu (2012).

However, this study was designed to derive an explicit formula for counting the number of distinct fuzzy subgroups (N) of D_{2p^nq} with respect to new equivalence relation $\approx$, defined on the lattice of all fuzzy subgroups of the group $FL(D_{2p^nq})$. In particular, we solve a problem posed by Tarnuaceanu (2016). The enumeration technique used in this work is more elegant because it yields fewer distinct fuzzy subgroups compared to existing techniques known in literature , see Tarnuaceanu (2012), Murali and Makamba (2001 and 2003). However, this technique is very rigorous and prolonged. In deriving the explicit formula, we started by using existing results from literature to find and count subgroups of D_{2p^nq}. We then applied modified Burnside Lemma from Tarnauceanu (2016) in counting the equivalence classes. For some p and q, the patterns for sets of subgroups fixed by automorphism group of D_{2p^nq} was used to form a bijective correspondence with N. Then, the explicit formulae for N was derived and the result proven by induction.

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The paper is divided into three sections. Section two gives some definitions, preliminary results and an overview of the new equivalence as defined in Tarnuaceanu (2016). Finally in section three we give an explicit formula for counting the number of distinct fuzzy subgroups of D_{2p^nq} with respect to $\approx$.

2. Preliminaries

In this section, we give some definitions, preliminary results and an overview of equivalence relations introduced in Tarnuaceanu (2016).

2.1 Definition: Given an arbitrary non empty set X, a fuzzy set on X is a function from X to the unit interval $I:[0,1]$. That is $\mu: X \to [0,1]$. Let G be a group and $\mathrm{F}(G)$ be collection of all fuzzy subsets of G. An element μ of $\mathrm{F}(G)$ is said to be a fuzzy subgroup of G if it satisfies the following two conditions:

- $\mu(x, y) \geq min\{\mu(x), \mu(y)\}, \forall\, x, y \in G.$
- $\mu(x^{-1}) \geq \mu(x), \forall x \in G.$

In this situation we have $\mu(x^{-1}) = \mu(x),$ for any $x \in G,$ and $\mu(e) = max\mu(G) = \sup \mu(G).$

The set $FL(G)$ which consist of all fuzzy subgroups of G forms a lattice with respect to the usual ordering of fuzzy set inclusion called fuzzy subgroup lattice.

For any $\alpha \in [0,1],$ the level subset is defined by $\mu_\alpha = \{x \in G \mid \mu(x) \geq \alpha\}$. Thus, a fuzzy subset μ is a fuzzy subgroup of G iff its level subsets are subgroups of G.

2.2 An overview of the new equivalence relation by Tarnauceanu (2016)

Let G be a group and $FL(G)$ be the lattice subgroups of G. An action of $Aut(G)$ on $FL(G)$ is defined using the concept of group action and that action is well defined. This action induces an equivalence relation on $FL(G)$. It is also known that every fuzzy subgroup of G determines a chain of subgroups of G which ends in G. The action can be seen in terms of chains of subgroups of G. Denote by $\bar{C}$, the set consisting of all chains of subgroups of G terminated in G.Then the previous action of $Aut(G)$ on $FL(G)$ can be seen as action of $Aut(G)$ on $\bar{C}$ and the previous equivalence relation is seen as equivalence relation induced by this action. An equivalence relation is then defined on $\bar{C}$ in the following manner: for two chains $C_1: H_1 \subset H_2 \subset \cdots \subset H_m = G$ and $C_2: K_1 \subset K_2 \subset \cdots \subset K_n = G$ of $\bar{C}$, we put $C_1 \approx C_2$ iff $m = n$ and $\exists\ f \epsilon\ Aut(G)$ such that $f(H_i) = K_i,\ i = \overline{i, n}$. The orbit of a chain in this situation is given by $C \epsilon\ \bar{C}\ \{f(C) | f \epsilon\ Aut(G)\}$, while the set of chains in $\bar{C}$ that are fixed by an automorphism of $f(G)$ is $Fix_{\bar{C}}(f) = \{C\ \epsilon \bar{C} | f(C) = C\}$. By applying Burnside's lemma, the number of distinct fuzzy subgroups N of G is given the equality

$$N = \frac{1}{|Aut(G)|} \sum_{f \in Aut(G)} \left| Fix_{\bar{C}}(f) \right| \qquad (1)$$

3. Explicit Formula for Counting Distinct Fuzzy Subgroups of D_{2p^nq}

It is well known in literature that dihedral groups D_{2p^nq} of order $2p^nq$, posses two subgroup structures, one of which is cyclic and isomorphic to $\mathbb{Z}_r$ and is of the form

$H_0^r = \langle a^{\frac{p^nq}{r}} \rangle$ $n = 2,3,5,6,10,15$ and $30.$, where r is a divisor of p^nq and the other $\frac{p^nq}{r}$

dihedral subgroups isomorphic to D_r, of the form $H_i^r = \langle a^{\frac{p^nq}{r}}, a^{i-1}b\rangle$, where $i = 1, 2, \cdots, \frac{p^nq}{r}$.
Our goal in this section is to give an explicit formula for counting the number of distinct fuzzy subgroups of D_{2p^nq}.

We start with the simplest case for class of dihedral groups being considered, that is, case $n=1, p=2$ and $q=3$. That is D_{12}, we have; $|Aut(D_{12})|=12$, $Aut(D_{2\times 2\times 3}) = Aut(D_{12}) = \{ f_{\alpha,\beta} \mid \alpha = \overline{0,5}$ *with* $(\alpha, 6) = 1,\ \beta = \overline{0,5} \}$
where $\overline{0,5}$ *means integer number from* 0 *to* 5.

Using Tarnuaceanu (2016), the subgroups of D_{12} were generated and are given below;

- $H_0^1 = \langle e \rangle$
- $H_0^k = \langle a^{\frac{6}{k}} \rangle$, *for* $k = 2, 3$ *and* 6. $n =$ 2,3,5,6,10,15 and 30.
- $H_k^1 = \langle a^{k-1}b \rangle$, *for* $k = 1,2,3,4,5$ *and* 6
- $H_k^2 = \langle a^3, a^{k-1}b \rangle$ *for* $k = 1, 2$ *and* 3.
- $H_k^3 = \langle a^2, a^{k-1}b \rangle$ *for* $k = 1$ *and* 2. $H_n^5 = \langle a^6, a^{n-1}b \rangle$, for $n = 1,2,3,4,5,6$.
- $H_1^6 = \langle a, b \rangle$.

The subgroups that are invariant with action of automorphism groups $Aut(D_{12})$ are given by;

Table 1: Subgroups fixed by $Aut(D_{12})$

α	β	$Fix(f_{\alpha,\beta})$
1	0	$\{H_0^1, H_0^2, H_0^3, H_0^6, H_1^1, H_2^1, H_3^1,, H_4^1, H_1^2, H_2^2, H_3^2, H_1^3, H_2^3, H_1^6\}$
1	1,5	$\{H_0^1, H_0^2, H_0^3, H_0^6, H_1^6\}$
1	2,4,	$\{H_0^1, H_0^2, H_0^3, H_0^6, H_1^3, H_2^3, H_1^6\}$
5	0	$\{H_0^1, H_0^2, H_0^3, H_0^6, H_1^1, H_4^1, H_1^2, H_1^3, H_2^3, H_1^6\}$
5	1	$\{H_0^1, H_0^2, H_0^3, H_3^2, H_0^6, H_1^6\}$
5	2	$\{H_0^1, H_0^2, H_0^3, H_0^6, H_2^1, H_5^1, H_2^2, H_1^3, H_2^3, H_1^6\}$
5	3	$\{H_0^1, H_0^2, H_0^3, H_0^6, H_1^2, H_1^6\}$
5	4	$\{H_0^1, H_0^2, H_0^3, H_0^6, H_3^1, H_6^1, H_1^3, H_2^3, H_3^2, H_1^6\}$
5	5	$\{H_0^1, H_0^2, H_0^3, H_0^6, H_2^2, H_1^6\}$

Now, computing from subgroup lattice from Figure 1, we have the following values:

- $|Fix_{\mathbb{C}}(f_{1,0})| = 68$
- $|Fix_{\mathbb{C}}(f_{1,1})| = |Fix_{\mathbb{C}}(f_{1,5})| = 12$
- $|Fix_{\mathbb{C}}(f_{1,2})| = |Fix_{\mathbb{C}}(f_{1,4})| = 20$

- $|Fix_{\bar{C}}(f_{1,3})| = 24$
- $|Fix_{\bar{C}}(f_{5,0})| = |Fix_{\bar{C}}(f_{5,2})| = |Fix_{\bar{C}}(f_{5,4})| = 36$
- $|Fix_{\bar{C}}(f_{5,1})| = |Fix_{\bar{C}}(f_{5,3})| = |Fix_{\bar{C}}(f_{5,5})| = 16$.

The number N of distinct fuzzy subgroups of D_{12} with respect to $\approx$ is given by the equality;

$$N = \frac{1}{12}[68 + (12\times 2) + (20\times 2) + 24 + (36\times 3) + (16\times 3)] = 26 \qquad (2)$$

Next we consider the case $p = 5, q = 3$ that is $D_{2\times 5\times 3} = D_{30}$. The subgroups of D_{30} are given by the following;

- $H_0^1 = \langle e\rangle$
- $H_0^k = \langle a^{\frac{15}{k}}\rangle$, *for* $k = 3, 5$ *and* 15. $n = 2,3,5,6,10,15$ and 30.
- $H_k^1 = \langle a^{k-1}b\rangle$, *for* $k = 1,2,3,4,5,6,7,8,9,10,11,12,13,14$ *and* 15
- $H_k^3 = \langle a^5, a^{k-1}b\rangle$ *for* $k = 1,2,3,4$ *and* 5.
- $H_k^5 = \langle a^3, a^{k-1}b\rangle$ *for* $k = 1,2$ *and* 3. $H_n^5 = \langle a^6, a^{n-1}b\rangle$, for $n = 1,2,3,4,5,6$.
- $H_1^{15} = \langle a, b\rangle$.

The order of the automorphism group of D_{30} is given by \$ $|Aut(D_{30})| = 15\times 8 = 120$. Next we find the subgroups that are fixed by each automorphisms of D_{30}.

Table 2: Subgroups fixed by $Aut(D_{30})$

α	β	$Fix(f_{\alpha,\beta})$
1	0	$\{All\ subgroups\}$
1	1,2,4,7,8,	
	11,13,14	$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}\}$
1	3,6,12	
1	5,10	
2,8,14	0	
2,8,14	1,7,13	$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_1^5, H_2^5, H_3^5\}$
2,8,14	2,14,11	
2,8,14	3,6,9	
2,8,142,8,14	4,13,7	$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_1^3, H_2^3, H_3^3, H_4^3, H_5^3\}$
2,8,14	5	$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_1^1, H_1^3, H_1^5\}$
2,8,14	6,12,3,	
2,8,14	7,4,1	
2,8,14	8,11,14	
2,8,14	9,3,12	$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_{15}^1, H_5^3, H_3^5\}$
2,8,14	10	
2,8,14	11,2,8,	$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_{14}^1, H_4^3, H_2^5\}$
2,8,14	12,9,6	$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_{13}^1, H_3^3, H_1^5\}$
2,8,14	13,0,0	

4,7,13	14,0,0	$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_{12}^1, H_2^3, H_3^5\}$
4,7,13	0,0,0	
4,7,13	3,6,12	$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_{11}^1, H_1^3, H_2^5\}$
4,7,13	6,12,9	
4,7,13	9,3,6	$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_{10}^1, H_5^3, H_1^5\}$
	12,9,3	
4,4,7,7,13,11	1,11,2,7,4,14	$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_9^1, H_4^3, H_3^5\}$
	2,7,4,14,8,13	
4,4,7,7,13,11	4,14,8,13,1,11	$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_8^1, H_3^3, H_2^5\}$
4,4,7,7,13,11	5,10,5,10,5,10	
4,4,7,7,13,11	8,13,1,11,2,7	$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_7^1, H_2^3, H_1^5\}$
4,4,7,7,13,11	2,8,14	
4,4,7,7,13,11	2,8,14	$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_6^1, H_1^3, H_3^5\}$
4,4,7,7,13,11	2,8,14	
4,4,7,7,13,11	2,8,14	$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_5^1, H_5^3, H_2^5\}$
4,4,7,7,13,11	0	
11		$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_4^1, H_4^3, H_1^5\}$
11	5	
11	10	$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_3^1, H_3^3, H_3^5\}$
11	1,4,7,13	$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_2^1, H_2^3, H_2^5\}$
	2,8,11,14	$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_1^1, H_6^1, H_{11}^1, H_1^3, H_1^5, H_2^5, H_3^5\}$
11	3,6,9,12	
11		$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_5^1, H_{10}^1, H_{15}^1, H_5^3, H_1^5, H_2^5, H_3^5\}$
11		
		$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_4^1, H_9^1, H_{14}^1, H_4^3, H_1^5, H_2^5, H_3^5\}$
		$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_3^1, H_8^1, H_{13}^1, H_3^3, H_1^5, H_2^5, H_3^5\}$
		$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_2^1, H_7^1, H_{12}^1, H_2^3, H_1^5, H_2^5, H_3^5\}$
		$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_2^1, H_7^1, H_{12}^1, H_4^3\}$
		$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_2^1, H_7^1, H_{12}^1, H_2^3\}$
		$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_2^1, H_7^1, H_{12}^1, H_3^3\}$
		$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_2^1, H_7^1, H_{12}^1, H_1^3\}$
		$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_2^1, H_7^1, H_{12}^1, H_5^3\}$
		$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_9^1, H_4^3, H_3^5\}$
		$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_9^1, H_4^3, H_3^5\}$
		$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_9^1, H_4^3, H_3^5\}$

	$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_9^1, H_4^3, H_3^5\}$
	$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_1^1, H_4^1, H_7^1, H_{10}^1, H_{13}^1, H_1^3, H_2^3, H_3^3, H_4^3, H_5^3, H_1^5\}$
	$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_2^1, H_5^1, H_8^1, H_{11}^1, H_{14}^1, H_1^3, H_2^3, H_3^3, H_4^3, H_5^3, H_2^5\}$
	$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_2^1, H_5^1, H_8^1, H_{11}^1, H_{14}^1, H_1^3, H_2^3, H_3^3, H_4^3, H_5^3, H_2^5\}$
	$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_3^1, H_6^1, H_9^1, H_{12}^1, H_{15}^1, H_1^3, H_2^3, H_3^3, H_4^3, H_5^3, H_3^5\}$
	$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_2^5\}$
	$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_3^5\}$
	$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_2^5\}$
	$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_1^5\}$

Now, computing subgroup lattice from Figure 2, we have the following values:

- $|Fix_{\bar{C}}(f_{1,0})| = 134$
- $|Fix_{\bar{C}}(f_{1,1})| = |Fix_{\bar{C}}(f_{1,2})| = |Fix_{\bar{C}}(f_{1,4})| = |Fix_{\bar{C}}(f_{1,7})| = |Fix_{\bar{C}}(f_{1,8})|$
  $= |Fix_{\bar{C}}(f_{1,11})| = |Fix_{\bar{C}}(f_{1,13})| = |Fix_{\bar{C}}(f_{1,14})| = 12$
- $|Fix_{\bar{C}}(f_{1,3})| = |Fix_{\bar{C}}(f_{1,6})| = |Fix_{\bar{C}}(f_{1,9})| = |Fix_{\bar{C}}(f_{1,12})| = 24$
- $|Fix_{\bar{C}}(f_{1,5})| = |Fix_{\bar{C}}(f_{1,10})| = 32$

- $|Fix_{\bar{C}}(f_{2,0})|=|Fix_{\bar{C}}(f_{2,1})|=|Fix_{\bar{C}}(f_{2,2})|=|Fix_{\bar{C}}(f_{2,3})|=|Fix_{\bar{C}}(f_{2,4})|=|Fix_{\bar{C}}(f_{2,5})|=$
  $|Fix_{\bar{C}}(f_{2,6})|=|Fix_{\bar{C}}(f_{2,7})|=|Fix_{\bar{C}}(f_{2,8})|=|Fix_{\bar{C}}(f_{2,9})|=|Fix_{\bar{C}}(f_{2,10})|=$
  $|Fix_{\bar{C}}(f_{2,11})|=|Fix_{\bar{C}}(f_{2,12})|=|Fix_{\bar{C}}(f_{2,13})|=|Fix_{\bar{C}}(f_{2,14})|=|Fix_{\bar{C}}(f_{8,0})|=$
  $|Fix_{\bar{C}}(f_{8,1})|=|Fix_{\bar{C}}(f_{8,2})|=|Fix_{\bar{C}}(f_{8,3})|=|Fix_{\bar{C}}(f_{8,4})|=|Fix_{\bar{C}}(f_{8,5})|=$
  $|Fix_{\bar{C}}(f_{8,6})|=|Fix_{\bar{C}}(f_{8,7})|=|Fix_{\bar{C}}(f_{8,8})|=|Fix_{\bar{C}}(f_{8,9})|=|Fix_{\bar{C}}(f_{8,10})|=$
  $|Fix_{\bar{C}}(f_{8,11})|=|Fix_{\bar{C}}(f_{8,12})|=|Fix_{\bar{C}}(f_{8,13})|=|Fix_{\bar{C}}(f_{8,14})|=|Fix_{\bar{C}}(f_{14,0})|=$
  $|Fix_{\bar{C}}(f_{14,1})|=|Fix_{\bar{C}}(f_{14,2})|=|Fix_{\bar{C}}(f_{14,3})|=|Fix_{\bar{C}}(f_{14,4})|=|Fix_{\bar{C}}(f_{14,5})|=$
  $|Fix_{\bar{C}}(f_{14,6})|=|Fix_{\bar{C}}(f_{14,7})|=|Fix_{\bar{C}}(f_{14,8})|=|Fix_{\bar{C}}(f_{14,9})|=|Fix_{\bar{C}}(f_{14,10})|=$
  $|Fix_{\bar{C}}(f_{14,11})|=|Fix_{\bar{C}}(f_{14,12})|=|Fix_{\bar{C}}(f_{14,13})|=|Fix_{\bar{C}}(f_{14,14})|=26$

- $|Fix_{\bar{C}}(f_{5,4})|=36$.

- $Fix_{\bar{C}}(f_{4,0})|=|Fix_{\bar{C}}(f_{4,3})|=|Fix_{\bar{C}}(f_{4,6})|=|Fix_{\bar{C}}(f_{4,9})|=|Fix_{\bar{C}}(f_{4,12})|=|Fix_{\bar{C}}(f_{7,0})|$
  $=|Fix_{\bar{C}}(f_{7,3})|=|Fix_{\bar{C}}(f_{7,6})|=|Fix_{\bar{C}}(f_{7,9})|=|Fix_{\bar{C}}(f_{7,12})|=|Fix_{\bar{C}}(f_{13,0})|=|Fix_{\bar{C}}(f_{13,3})|$
  $=|Fix_{\bar{C}}(f_{13,6})|=|Fix_{\bar{C}}(f_{13,9})|=|Fix_{\bar{C}}(f_{13,12})|=46$

- $|Fix_{\bar{C}}(f_{11,0})|=|Fix_{\bar{C}}(f_{11,5})|=|Fix_{\bar{C}}(f_{11,10})|=66$

- $|Fix_{\bar{C}}(f_{4,1})|=|Fix_{\bar{C}}(f_{4,2})|=|Fix_{\bar{C}}(f_{4,4})|=|Fix_{\bar{C}}(f_{4,5})|=|Fix_{\bar{C}}(f_{4,7})|=|Fix_{\bar{C}}(f_{4,8})|=$
  $|Fix_{\bar{C}}(f_{4,10})|=|Fix_{\bar{C}}(f_{4,11})|=|Fix_{\bar{C}}(f_{4,13})|=|Fix_{\bar{C}}(f_{4,14})|=|Fix_{\bar{C}}(f_{7,1})|=|Fix_{\bar{C}}(f_{7,2})|=$
  $|Fix_{\bar{C}}(f_{7,4})|=|Fix_{\bar{C}}(f_{7,5})|=|Fix_{\bar{C}}(f_{7,7})|=|Fix_{\bar{C}}(f_{7,8})|=|Fix_{\bar{C}}(f_{7,10})|=|Fix_{\bar{C}}(f_{7,11})|=$
  $|Fix_{\bar{C}}(f_{7,13})|=|Fix_{\bar{C}}(f_{7,14})|=|Fix_{\bar{C}}(f_{11,1})|=|Fix_{\bar{C}}(f_{11,2})|=|Fix_{\bar{C}}(f_{11,3})|=|Fix_{\bar{C}}(f_{11,4})|=$
  $|Fix_{\bar{C}}(f_{11,6})|=|Fix_{\bar{C}}(f_{11,7})|=|Fix_{\bar{C}}(f_{11,8})|=|Fix_{\bar{C}}(f_{11,9})|=|Fix_{\bar{C}}(f_{11,11})|=|Fix_{\bar{C}}(f_{11,12})|=$
  $|Fix_{\bar{C}}(f_{11,13})|=|Fix_{\bar{C}}(f_{11,14})|=|Fix_{\bar{C}}(f_{13,1})|=|Fix_{\bar{C}}(f_{13,2})|=|Fix_{\bar{C}}(f_{13,4})|=|Fix_{\bar{C}}(f_{13,5})|=$
  $|Fix_{\bar{C}}(f_{13,7})|=|Fix_{\bar{C}}(f_{13,8})|=|Fix_{\bar{C}}(f_{13,10})|=|Fix_{\bar{C}}(f_{13,11})|=|Fix_{\bar{C}}(f_{13,13})|=|Fix_{\bar{C}}(f_{13,14})|=16$

From the computation above we have that; the number N of distinct fuzzy subgroups of D_{30} with respect to $\approx$ is given by the equality;

$$N=\frac{1}{120}[134+(12\times 8)+(24\times 4)+(32\times 2)+(26\times 3\times 15)+$$

$$+(46\times 3\times 5)+(16\times 3\times 10)+(66\times 3)+(16\times 12)]=26 \qquad (3)$$

From lattice subgroup structure of D_{2pq}, we have that the number N of distinct fuzzy subgroups of D_{2pq} with respect to $\approx$ is 26. For example, $D_{2\times 2\times 5}, D_{2\times 3\times 5}, D_{2\times 7\times 2}$ each has 26 distinct fuzzy subgroups. Similarly, we compute N for the case $n=2$. That is D_{2p^2q}. For example $D_{2\times 2^2\times 7}, D_{2\times 3^2\times 2}, D_{2\times 5^2\times 3}$ has 88 distinct fuzzy subgroups with respect to $\approx$. More generally,

Table 3: Distinct Fuzzy Subgroups of Dihedral groups of Order $2p^nq$,for some p , q and n

	N	p	q	N
$D_{2\times 2\times 7}, D_{2\times 3\times 2}, D_{2\times 5\times 3}$	1	2,3,5	7,2,3	26
$D_{2\times 2^2\times 7}, D_{2\times 5^2\times 2}, D_{2\times 11^2\times 3}$	2	2,5,11	7,2,3	88
$D_{2\times 2^3\times 7}, D_{2\times 3^3\times 2}, D_{2\times 5^3\times 3}$	3	2,3,5	7,2,3	264
$D_{2\times 2^3\times 7}, D_{2\times 3^3\times 2}, D_{2\times 5^3\times 3}$	4	2,3,5	7,2,3	736

3.1 Theorem: *The number N of all distinct fuzzy subgroups of dihedral group of order* $2p^nq$ *, that is* D_{2p^nq} *is given by the equality* $N = 2^n[\mathrm{n}(\mathrm{n}+6)+6]$.

Proof: Firstly, we assume without loss of generality that $p=5,\ q=3$. We notice that there is one-to-one correspondence between the number of distinct fuzzy subgroups of $D_{2(p^nq)} = D_{2\times 5^n\times 3}$ and the value $|Fix_{\overline{C}}(f_{2,\beta})|$ *for any* $\beta = 0,(5^n\times 3)-1$. We now prove by induction on n.

The statement is clearly true for $n=1,2$ *and* 3, that is $D_{2\times 5^1\times 3} = D_{30}$ has 26 distinct fuzzy subgroups, $D_{2\times 5^2\times 3} = D_{150}$ has 88 distinct fuzzy subgroups and $D_{2\times 5^3\times 3}$ has 264 distinct fuzzy subgroups.

Assume the statement is true for n=k, that is $D_{2\times 5^k\times 3}$ has $2^k[k(k+6)+6]$ distinct fuzzy subgroups, then set $\beta = 0$. The set of subgroups fixed for *n*=*k* for the element of automorphism group $f_{2,0}$ is given by

$$Fix(f_{2,0})_k = \{H_0^1, H_1^1, H_0^5, H_1^5, H_0^{5^2}, H_1^{5^2}, H_0^{5^3}, H_1^{5^3}, \cdots, H_0^{5^k}, H_1^{5^k}, H_0^3, H_1^3, H_0^{5\times 3}, H_1^{5\times 3}, H_0^{5^2\times 3}, H_1^{5^2\times 3}, H_0^{5^3\times 3}, H_1^{5^3\times 3}, \cdots, H_0^{5^k\times 3}, H_1^{5^k\times 3}\}, \quad (4)$$

this in turn gives the corresponding value $|Fix_{\overline{C}}(f_{2,\beta})| = 2^k[k(k+6)+6]$.

Next we show that $D_{2(5^{k+1}\times 3)}$ has $2^{k+1}[(k+1)(k+1+6)+6]$ distinct fuzzy subgroups. The set of subgroups that are fixed by the element of automorphism group $f_{2,0}$ for *n*=*k+1* is given by

$$Fix(f_{2,0})_{k+1} = \{H_0^1, H_1^1, H_0^5, H_1^5, H_0^{5^2}, H_1^{5^2}, H_0^{5^3}, H_1^{5^3}, \cdots, H_0^{5^k}, H_1^{5^k}, H_0^{5^{k+1}}, H_1^{5^{k+1}} H_0^3, H_1^3, H_0^{5\times 3}, H_1^{5\times 3}, H_0^{5^2\times 3}, H_1^{5^2\times 3}, H_0^{5^3\times 3}, H_1^{5^3\times 3}, \cdots, H_0^{5^k\times 3}, H_1^{5^k\times 3}, H_0^{5^{k+1}\times 3}, H_1^{5^{k+1}\times 3}\}. \quad (5)$$

From the inductive hypothesis, observe that $Fix(f_{2,0})_{k+1}$ has four(4) more subgroups than $Fix(f_{2,0})_k$. From the lattice subgroup structure of $D_{2\times 5^{k+1}\times 3}$ these four(4) subgroups yields a further $2^k[k^2+10k+20]$ distinct fuzzy subgroups. Adding together all distinct fuzzy subgroups we have a total given by the equality

$$N = |Fix_{\bar{C}}(f_{2,0})| = 2^k[k(k+6)+6] + 2^k[k^2+10k+20]. \qquad (6)$$

Hence, this completes the induction □.

4. Conclusion

Dihedral groups of this form have similar lattice subgroup structures, thus they yield the same number of distinct fuzzy subgroups with respect to the equivalence relation $\approx$. However, computation with respect to the equivalence relation $\approx$ can be extended to other classes of nonabelian groups.

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