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# An Extended Study of Motion in the Vicinity of Non Static Spherical Gravitational Potential Fields

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## Abstract
*In this article, the generalized Schwarzschild metric for time varying spherical fields is used to derive equations of motion for test particles in the vicinity of a time varying spherical mass. A generalized gravitational scalar potential is expanded and used to construct equations of motion for test particles exterior to a time varying spherical distribution of mass. The world line element at the equatorial plane of a spherical massive body was constructed. This was used to study the motion of photons. The equations of motion obtained have additional terms not found in equivalent equations in the Schwarzschild's field. These are uncovered for theoretical and astrophysical development and applications. Remarkably, when the time varying scalar gravitational potential reduces to that of a static field, results obtained in Schwarzschild's field are obtained. Thus, our generalization is mathematically satisfactory and has astrophysical implications for the study of time varying spherical gravitational fields.*

**Keywords:** Motion, Test Particles, Photons, Planets, Mass, Non Static

## Introduction
In the year 1915, Albert Einstein published his geometrical laws of gravitation which is widely accepted as one of the major theories of gravitation. The first part is the geometrical law of the gravitational field, popularly called the Einstein's field equations. The second part is the geometrical law of motion for test particles of nonzero rest masses and photons (Weinberg, 1972, Howusu 2010). The exact solution to these field equations was constructed by Schwarzschild for a static homogeneous spherical mass distribution for integration, interpretation and application to the motions of test particles (Howusu 2010). In Schwarzschild metric, the tensor field varies with only the radial distance. It is the metric tensor exterior to static spherically symmetric body situated in an empty space (Howusu 2003, Lumbi *et a*l., 2014). Research has shown that some of the astrophysical bodies such as the Sun and other stars are not perfectly spherical, their field cannot depend on only the radial distance as assumed by Schwarzschild (Howusu 2003).

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The well - known Schwarzschild's metric in spherical polar coordinates is given as (Weinberg, 1972, Howusu, 2007, 2010)

$$g_{00} = 1 + \frac{2f(r)}{c^2} \tag{1.1}$$

$$g_{11} = -\left[1 + \frac{2f(r)}{c^2}\right]^{-1} \tag{1.2}$$

$$g_{22} = -r^2 \tag{1.3}$$

$$g_{33} = -r^2 \sin^2\theta \tag{1.4}$$

with

$$f(r) = \frac{GM}{r} \tag{1.5}$$

where the radial distance, $r > R$, the radius of the static spherical mass, $G$ is the universal gravitational constant, $M$ is the total mass of the distribution and $c$ is the speed of light in vacuum.

The covariant metric tensors exterior to a homogeneous time varying distribution of mass within regions of spherical geometry has been deduced as (Weinberg, 1972, Howusu, 2007, Izam & Jwanbot, 2013 and Ndikilar, 2014)

$$g_{00} = 1 + \frac{2}{c^2} f(t,r) \tag{1.6}$$

$$g_{11} = -\left[1 + \frac{2}{c^2} f(t,r)\right]^{-1} \tag{1.7}$$

$$g_{22} = -r^2 \tag{1.8}$$

$$g_{33} = -r^2 sin^2\theta \tag{1.9}$$

where $f(t,r)$ is a function dependent on the mass distribution within the sphere that experiences radial displacement and has been obtained to be given approximately as (Ndikilar, *et., al* 2008)

$$f(t,r) \approx -\frac{k}{r} \exp\left[i\omega\left(t - \frac{r}{c}\right)\right] \tag{1.10}$$

where $\omega$ is the angular velocity of the mass distribution within the sphere about its axis of symmetry. The contravariant metric tensor, affine connections, curvature tensor, Ricci tensor and field equations for this distribution have been derived (Ndikilar, *et., al* 2008 and Ndikilar, 2014).

In this article, we obtain explicit expressions for motion of test particles and photons in this gravitational field using equation (1.10) and all related relativistic parameters derived in the aforementioned research articles.

## Methodology

The generalized Schwarzschild metric for time varying spherical fields were used to derive equations of motion for test particles in the vicinity of a time varying spherical mass. The equation of motion for test particles and the gravitational scalar potential for time varying spherical mass distribution were applied to derive the variation of the time on a clock moving in this field and hence the planetary equation of motion in the equatorial plane of this gravitational field.

## Results and Discussions

### Motion of Test Particles

The relativistic equation of motion for test particles is given explicitly as (Weinberg, 1972, Howusu, 2007, Ndikilar, 2008, Ndikilar and Howusu 2010, Lumbi *et., al 2014*)

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^{\mu}_{\nu\lambda}\left(\frac{dx^\nu}{d\tau}\right)\left(\frac{dx^\lambda}{d\tau}\right) = 0 \tag{2.1}$$

where the indices run from 0 to 3 and the symbols have their usual meaning. The time equation of motion is obtained by setting $\mu = 0$ in equation (2.1) and substituting the appropriate affine connections. This gives

$$c\ddot{t} + \frac{1}{c}\left[1+\frac{2}{c^2}f\right]^{-1}\frac{\partial f}{\partial t}\dot{t}^2 + \frac{2}{c^2}\left[1+\frac{2}{c^2}f\right]^{-1}\frac{\partial f}{\partial r}\dot{t}\dot{r}$$
$$-\frac{1}{c^2}\left[1+\frac{2}{c^2}f\right]^{-3}\frac{\partial f}{\partial t}\dot{r}^2 = 0 \tag{2.2}$$

where the dot denotes differentiation with respect to proper time.

Dividing equation (2.2) all through by $c\dot{t}$ and neglecting terms in $c^{-3}$ gives

$$\frac{\ddot{t}}{\dot{t}} + \frac{2}{c^2}\left[1+\frac{2}{c^2}f\right]^{-1}\frac{\partial f}{\partial t}\dot{r} + \frac{1}{c^2}\left[1+\frac{2}{c^2}f\right]^{-1}\frac{\partial f}{\partial t}\dot{t} = 0 \tag{2.3}$$

Equation (2.3) can be written approximately as

$$\frac{d}{d\tau}\left\{\ln\dot{t} + \ln\left(1+\frac{2}{c^2}f\right)\right\} = 0 \tag{2.4}$$

Integrating equation (2.4) and solving we have

$$\dot{t} = A\left(1+\frac{2}{c^2}f\right)^{-1} \tag{2.5}$$

where $A \equiv 1$ is the constant of integration since as $t \to \tau,\ f(t,r) \to 0$ ..

Equation (2.5) is the expression for the variation of the time on a clock moving in this gravitational field. It is similar in form to that of Schwarzschild's gravitational field but differs only in the expression of $f$ in the two fields.

Now consider the gravitational scalar potential function $f(t,r)$ for a time varying spherical mass distribution is given in equation (1.10) and let

$$x = \omega\left(t - \frac{r}{c}\right) \tag{2.6}$$

Thus equation (1.10) can be written as

$$f(t,r) \approx -\frac{k}{r}e^{ix} \tag{2.7}$$

Using the series expansion relation

$$e^{ix} = 1 - \frac{x^2}{2} + ix + .... \tag{2.8}$$

gives an approximate expression of equation (2.7) as

$$f(t,r) \approx -\frac{k}{r}\left(1 - \frac{x^2}{2} + ix + ...\right) \tag{2.9}$$

Neglecting higher powers of terms in equation (2.9) and taking the modulus of $f(t,r)$ gives

$$\left|f(t,r)\right| \approx \left\{\frac{k^2}{r^2}\left[\left(1 - \frac{x^2}{2}\right)^2 + x^2\right]\right\}^{\frac{1}{2}} \tag{2.10}$$

Simplifying equation (2.10), expanding binomially and neglecting the higher powers of $x$, gives

$$f(t,r) \approx \frac{k}{r}\left[1 + \frac{\omega^4}{8}\left(t - \frac{r}{c}\right)^4\right] \tag{2.11}$$

It is profound to note that equation (2.11) reduces to the gravitational potential for static Schwarzschild's field when the angular velocity $\omega = 0$ and thus gives an extension of Newton's gravitational scalar potential to time varying spherical mass distributions.

An explicit expression for the time equation of motion for test particles of non-zero rest mass (2.5) can thus be written explicitly using equation (2.11) as

$$\dot{t} = \left\{1 + \frac{2k}{c^2 r}\left[1 + \frac{\omega^4}{8}\left(t - \frac{r}{c}\right)^4\right]\right\}^{-1} \tag{2.12}$$

Remarkably, this equation (2.12) reduces to Schwarzschild’s time equation when $f(t,r)$ reduces to $f(r)$. The term $\frac{k\omega^4}{4c^2r}\left(t-\frac{r}{c}\right)^4$ is an additional term in the time equation of motion which does not appear in Schwarschild's field. It clearly captures the time dependence of the field and the rotational effect of the mass distribution on the motion. It has great consequences to the motion of test particles in this field and is thus uncovered for theoretical and astrophysical development and applications. This is distinct from other time equations of motion for test particles obtained earlier for other fields (Ndikilar, 2009 and Ndikilar, *et., al* 2008)

Similarly, set $\mu = 1$ in equation (2.1) and substituting relevant affine connection coefficients gives the radial equation of motion as

$$\ddot{r}+\left(1+\frac{2}{c^2}f\right)\frac{\partial f}{\partial r}\dot{t}^2-\frac{2}{c}\left(1+\frac{2}{c^2}f\right)^{-1}\frac{\partial f}{\partial t}\dot{t}\dot{r}-\frac{1}{c^2}\left(1+\frac{2}{c^2}f\right)^{-1}\frac{\partial f}{\partial r}\dot{r}^2$$
$$-r\left(1+\frac{2}{c^2}f\right)\dot{\theta}^2-r\sin^2\theta\left(1+\frac{2}{c^2}f\right)\dot{\phi}^2=0 \quad (2.13)$$

For pure radial motion, $\dot{\theta}\equiv\dot{\phi}=0$ and using (2.5) reduces equation (2.13) to

$$\ddot{r}+\left(1+\frac{2}{c^2}f\right)^{-1}\left\{1-\frac{2}{c^2}\left(1+\frac{2}{c^2}f\right)^{-1}\dot{r}-\frac{\dot{r}^2}{c^2}\right\}\frac{\partial f}{\partial r}=0 \quad (2.14)$$

Expanding equation (2.14), neglecting higher terms of $c^{-2}$ and substituting for $f$ from equation (2.11) gives

$$\ddot{r}-\frac{1}{c^2}\dot{r}^2-\frac{2}{c^2}\dot{r}+1-\frac{2k}{c^2r}\left[1+\frac{\omega^4}{8}\left(t-\frac{r}{c}\right)^4\right]=0 \quad (2.15)$$

which is the pure radial equation of motion for particles of non-zero rest masses in this gravitational field. This equation also reduces to Schwarzschild’s pure radial equation of motion if the mass distribution is static.

Also, setting $\mu = 2$ into equation (2.1) and substituting relevant affine connection coefficients yields

$$\ddot{\theta}+\frac{2}{r}\dot{r}\dot{\theta}+\frac{1}{2}\dot{\phi}^2\sin 2\theta=0 \quad (2.16)$$

This is the polar equation of motion for a test particle in the vicinity of a time varying spherical mass distribution.

Similarly, setting $\mu = 3$ into equation (2.1) yields

$$\ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} + 2\dot{\theta}\dot{\phi}\cot\theta = 0 \tag{2.17}$$

Equation (2.17) is the azimuthal equation of motion for a particle of non zero rest masses in Schwarzschild's field.

This equation is equal to the azimuthal equation of motion for a particle of non-zero rest mass in the Schwarzschild's field. Therefore, the instantaneous azimuthal angular velocity from our field is exactly the same as that obtained from Newton's theory of gravitation and Schwarzschild's metric (Ndikilar *et. al,* 2008).

For motion along the equatorial plane, it can be shown that equation (2.17) reduces to

$$r^2\dot{\phi} = l \tag{2.18}$$

where, $l$ is a constant which is physically equivalent to the angular momentum of test particles in this gravitational field. This is equivalent to that obtained in Schwarschild's gravitational field. We can deduce that the law of conservation of angular momentum is invariant in the two gravitational fields.

**Motion of Photons along the Equatorial Plane**

The invariant world line element in the exterior region of this time varying homogenous spherical body is given by

$$c^2 d\tau^2 = c^2\left[1 + \frac{2}{c^2} f(t,r)\right]dt^2 - \left[1 + \frac{2}{c^2} f(t,r)\right]^{-1} dr^2 - r^2 d\theta^2 - r^2\sin^2\theta d\phi^2 \tag{3.1}$$

Relativistic mechanics demands that light (photon) should move along a null geodesic; that is

$$c^2 d\tau^2 = 0 \tag{3.2}$$

Thus, for a photon, equation (3.1) reduces to:

$$0 = c^2\left[1 + \frac{2}{c^2} f(t,r)\right]dt^2 - \left[1 + \frac{2}{c^2} f(t,r)\right]^{-1} dr^2 - r^2 d\theta^2 - r^2\sin^2\theta d\phi^2 \tag{3.3}$$

Suppose the photon moves along the equatorial plane of the time varying homogenous spherical mass, ($\theta = \frac{\pi}{2}$); then equation (3.3) reduces to;

$$0 = c^2\left[1 + \frac{2}{c^2} f(t,r)\right]dt^2 - \left[1 + \frac{2}{c^2} f(t,r)\right]^{-1} dr^2 - r^2 d\phi^2 \tag{3.4}$$

Dividing (3.4) all through by $d\tau^2$ gives

$$0=\left(1+\frac{2}{c^2}f(t,r)\right)\dot{t}^2+\left(1+\frac{2}{c^2}f(t,r)\right)^{-1}\dot{r}^2+r^2\dot{\phi}^2 \tag{3.5}$$

Substituting equation (2.5) and (2.18) into equation (3.5) and simplifying yields

$$\dot{r}^2+\left(1+\frac{2}{c^2}f(t,r)\right)\frac{l^2}{r^2}+1=0 \tag{3.6}$$

Now, let $u(\phi)=\dfrac{1}{r(\phi)}$ , then

$$\dot{r}=-l\frac{du}{d\phi} \tag{3.7}$$

Substituting equation (3.7) in equation (3.6) and simplifying gives

$$\left(\frac{du}{d\phi}\right)^2+\left(1+\frac{2}{c^2}f)\right)u^2+\frac{1}{l^2}=0 \tag{3.8}$$

Differentiating equation (3.8), we have

$$\frac{d^2u}{d\phi^2}+u\left[\left(1+\frac{2}{c^2}f\right)+\frac{u}{c^2}\frac{\partial f}{\partial u}\right]\frac{du}{d\phi}=0 \tag{3.9}$$

This is the planetary equation of motion in the equatorial plane of this gravitational field. This equation has additional terms (due to the time component of the mass distribution), not found in the corresponding equation in Schwarzschild's field.

**Conclusion**

The generalized Schwarzschild metric tensor was used to construct the equations of motion for test particles and photons in the gravitational field exterior to a time varying spherical mass distribution. The time, radial, polar and azimuthal equations of motion for particles of non-zero rest masses in this gravitational field are given by equations (2.2), (2.15), (2.16) and (2.17) respectively while the planetary equation of motion in the equatorial plane of this gravitational field is given by (3.9). These equations contain additional terms not found in the famous Schwarzschild's field which are open up for theoretical development and applications.

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