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# Optimal Stochastic Control of Linear Systems with State and Control Dependent Noise: Efficient Computational Algorithms

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## Abstract

*In this paper two new numerically efficient algorithms to solve the state and control–depended noise linear–quadratic optimal control problem are proposed. The formulation extends the Hoskins-Walton method for square-root matrix computations and some other recent results for deterministic algebraic Riccati equations resolution. The Implementation require as basic operations: additions, products and inversions in the field of positive definite matrices. The approach is quite general since it can be applied for solving optimal control problems associated to processes with multiple independent noises in the control and state vectors.*

Keywords: **Noise, Linear-Quadratic, Definite Matrices, Control and State Vectors**

## 1. INTRODUCTION

Consider the stochastic linear system formally described by the stochastic Itồ equation (Kleinman, 1999, McLane, 2001).

$d_x(t) = A_x(t)dt + B_u(t)dt + F_x(t)d\alpha + G_u(t)d\beta + dw$……………………………………..(1)

and

$y(t) = C_x(t)$……………………………………………………………………….(2)

Where A, and F are constant n x nmatrices, B and G are constant n x m matrices. C is an rxn constant matrix, while F and G have full rank. The pair <A,B> can be stabilized, while the pair<A,C> is completely detectable. Moreover w(t), α(t), β(t) are independent Wiener processes with autocorrelations.

$E\{[w(t) – w(\tau)][w(t) – w(\tau)]^T\} = W/t-\tau/$………………………………………………..(3)

$E\{[\alpha (t) – \alpha(\tau)]^2\} = \sigma a - \tau/$…………………………………………………………….(4)

$E\{[\beta (t) –\beta(\tau)]^2\} = \rho/t - \tau/$……………………………………………………………….(5)

Now, if we define the steady–state cost:

$J = E\{y^T(t)y(t) + u^T(t)R_u(t)\}$………………………………………………………..(6)

where R = $R^T$>0. It is well known (Mclane, 2001) that the optimal linear feedback control that minimizes J is given by:

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$$\mu_0(t) = -(R + \rho G^T K G)^{-1} B^T K_x(t) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

Where$k$ is the unique positive definite solution (where it exists) of the Stochastic AlgebraicRiccati Equation (SARE):

$$L + A^T K + KA + \sigma F^T K F - KB(R + \sigma G^T K G)^{-1} B^T K = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

where L =$L^T C^T C$ $\geq 0$. From the terms $\sigma$F$^{\mathrm{T}}$ KF and $\rho$G$^{\mathrm{T}}$ KG this equation is commonly known as the Deterministic Algebraic Riccati Equation (DARE).

Several numerical technical techniques have been investigated in the literature for solving Equation (8). Most of the said techniquesare natural extensions of the Kleinmaniterative method for deterministic ARE problems (Kleinman, 2006). To summarize this computationally efficient method, let us consider Equation (8) and the following iterative procedure:

**Algorithm 1**: if a solution k > 0 of the SARE in Equation (8) exists, then k =$lim K_i$, where $K_i$> 0, i = 1,2,... is the unique positive definite solution of the ARE.

$$A^T K_i + K_i A - K_i B(R + \rho G^T K_{i-1} G)^{-1} B^T K_{t-\sigma} F^T K_{i-1} F + L = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9)$$

where $K_0$= 0. If a solution K > 0 does not exists then, $K_i$ diverges. Hence, the technique advocated by Kleinman consists in solving the SARE as a limiting sequence of solutions of conventional ARE and,the solution of each ARE is obtained as a limiting sequence of solutions of linear matrix Liapunov type equations (Kleinman, 2008). The proof of the algorithm convergence is bounded based on the fact that ARE solution is a monotone function.

This is in the positive definite matrix norm sense of the additive term of $\sigma F^T K_{i-1} F$ being also monotone with respect to $K_i$= R + $\rho$G$^{\mathrm{T}}$$K_{i-1}$G. Hence, if $k$ exists, then sequence $K_i$, i = 1,2...is bounded above and as such we have convergence. By using this procedure jointly with an iterative algorithm to solve deterministic ARE for each $K_i$, it is advisable to initialize this algorithm by utilizing the last computed $K_{i-1}$. The strategy leads to a drastic reduction in the number of linear equations needed to compute $K_i$from $K_{i-1}$and has been reported by Kleinman (2006).

In this paper, some previous results on square root matrix computations (Hoskins and Walton, 2008) and on deterministic ARE computations (Incertis and Martinez, 2007; Incertis, 2001, 2002) are generalized to the stochastic case and two new numerical methods for the solution of SARE are presented. Application of both algorithms requires computation of a sequence of positive definite matrices by means of an iterative procedure that makes use of matrix additions, products and inversions. The results are particularly useful when the system matrix say A is a near-diagonal or diagonal dominant matrix. This is particularly the case when dealing with many spatially quantized distributed parameter systems. For this class of systems the proposed numerical algorithms will converge in a few program rounds.

Another advantage of the proposed approach is that all computations are performed in the field of the positive definite matrices. This require that we resort to complex number arithmetic as occurs in most currently available efficient methods (Laub, 1979; Larinet al., 1981).The structure of this paper is as follows: Section 2 presents a reformulation of the SARE problem and some previous results and theorems which are useful to the development of new computational algorithms. Section 3 is devoted to the presentation of the two new algorithms and the corresponding convergence conditions. In Section 4, an analysis of the computational cost and storage requirements of the main algorithm is performed. The resultsare compared againstother relatedand efficient methods. In section 5, an example is solved to show the behaviour of Algorithm 2. Finally, Section 6 summarizes the main contribution and features of the proposed algorithms in this paper.

## 2. METHODOLOGY

Weassume that the matrix C is an n x n matrix and invertible, hence $L = C^T C$will be a positive definite symmetric matrix. Under the fulfillment of the existence and uniqueness conditions, the unique positive definite solution K of Equation(8) must be computed. Now, under those assumptions, K$^{-1}$ is also unique and positive definite.The pre-and post-multiplication of Equation 8by K$^{-1}$ then gives:

$K^{-1}LK^{-1} + K^{-1}A^T + AK^{-1} + \sigma K^{-1}F^T KFK^{-1} - B(R + \rho G^T KG)^{-1}BT = 0$……………….(10)

The pre-multiplication of this equation by matrix L leads to:
$LK^{-1}LK^{-1} + LK^{-1}A^T + LAK^{-1} + \sigma LK^{-1}F^T KFK^{-1} - LB(R + \rho G^T KG)^{-1}BT = 0$………..(11)

This may be factored out as:
$(LK^{-1})(LK^{-1} + A^T) - (A - L - L^{-1})LK^{-1} = KG)^{-1}BT + AA^T - \sigma LK^{-1}F^T KFK^{-1}$……...(12)

Now, introducing the notation:
$X = LK^{-1} + A^T$…………………………………………………..……(13)

After a straight forward computation, Equation (8) becomes that offindinga matrix X such that:
$X^2 + (LAL^{-1} - A^T)X =$
$LAL^{-1}A^T \div LB[R + \rho G^T(X - A^T)^{-1}LG]^{-1}B^T - \sigma(x - A^T)(X - A^T)^{-1}LFL^{-1}(X -$
$A^T)$……………………………………………………..…………..…..(14.1)
$X - A^T > 0$……………………………………………………..………………..(14.2)

Where, condition (14.2) is equivalent to imposing matrix K to be apositive definite matrix. Moreover, let by definition:
$2D = LAL^{-1}A^T$………………………………..………………………...(15.1)
$T(X) = LAL^{-1}A^T + LB[R - \sigma G^T(X - A^T)^{-1}LG]^{-1}B^T - \sigma(X - A^T)^{-1}F^T(X - A)^{-1}LFL^{-1}(X - A^T)$
…………………………………..……………………………….. (15.2)

By utilizing this notation Equation (14.1) can be written more compactly as:
$X^2 + 2DX = T(X)$………………………………..……………………........(16)

Thus, the problem becomes that of verifying and finding a solution to X from Equation (14.2), in order to formulate new algorithms for the stochastic optical control problem. Some previous results (Incertis, 1981) for deterministic algebraic Riccati equation are very much required.

## 2.1 Deterministic Matrix Riccati Equations:

In the deterministic case F and G are null matrices. Hence, Equation (8) takes the form:

$$L + A^T K + KA - KBR^{-1}B^T K = 0 \qquad (17)$$

Particularizing Equation (16) to this case, we can write:

$$X^2 + 2DX = T \qquad (18)$$

Where,

$$T = LAL^{-1}AT + LBR^{-1}B^T \qquad (19)$$

Furthermore, consider transformation:

$$H = X + D \qquad (20)$$

In Equation (18), a straight forward computation then gives:

$$H^2 + DH = HD = M \qquad (21)$$

where:

$$M = T + D^2 = LAL^{-1}A^T + LBR^{-1}B^T + D^2 \qquad (22)$$

Now, if M is a positive definite matrix, then a positive definite square root matrix $M^{\frac{1}{2}}$ exists and is unique. For instance, if M is semi simple, it may always be diagonalized in the form of:

$$M = P\Lambda P^{-1} \qquad (23)$$

where$\Lambda$> 0 is strictly diagonal. The square root matrix $M^{\frac{1}{2}}$is then given by:

$$M^{\frac{1}{2}} = (P\Lambda P^{-1})^{\frac{1}{2}} \qquad (24a)$$

or

$$M^{\frac{1}{2}}=P\Lambda^{\frac{1}{2}}P^{-1} > 0 \qquad (24b)$$

From the preceding formulation, the following fundamental theorem is demonstrated (Incertis, 2001):

**Theorem 1**: if [$M^{\frac{1}{2}}$, D]$\triangleq$ $M^{\frac{1}{2}}$D-D$M^{\frac{1}{2}}$ = 0, Then the unique solution of Equation (18) which fulfills the positive definiteness stated in condition (14.2) is given by:

$$M^{\frac{1}{2}} - D \qquad (25)$$

Notice that from Equation (13) the solutions of Equations (17) and (18) can be related by the following theorem, which is stated here without proof:

**Theorem 2**: if <A,B> is stabilizable, <A, C> is detectable, and L = CTC is a positive definite and a solution X of Equation (18) is obtained for which condition (14.2) is fulfilled, then the unique positive definite solution of the Algebraic Riccati Equation (ARE)in Equation (17) is given by:

$$X = (X - A^T)^{-1}L \qquad (26)$$

In order to apply this theoretical result to general ARE problems for which commutatively condition [$M^{\frac{1}{2}}$, D] = 0 is not identically satisfied, the following numerical algorithm has been

formulated (Incertis, 2002) as an extension of the Hoskins–Walton method for square–root matrix computations: ARE Algorithm is a solution X, where if Equation (18) exists, then
X = lim $X_i$, where $X_0$ = T and:

$$X_{i+1} = -D + (X_i + TX_i^{-1})\text{.........................................} \quad \text{...(27)}$$

For the algorithm to be useful, it is required that $X_i$; i = 0,1,2...be a convergent sequence of positive definite matrices. The following necessary and sufficient condition for the convergence of this iterative procedure is given as (Incertis, 2002):

$$\|\tau^{\frac{1}{2}} - D\| > / \sqrt{3}\,\|T^{\frac{1}{2}}\| > 0\text{.........................................................................}(28)$$

We shall note at this point that this algorithm can also be initialized by utilizing any good apriori estimates of the solution when available, such as $X_0 = T^{1/2}$, or some positive definite matrix $X_0$ for which $\|X_0^{1/2} - D\| > 0$. In the following section the previous theoretical and algorithmic results on deterministic Riccatis equations are generalized and applied to solve the more complex and difficult SARE problem.

## 3. SARE ALGORTHMS

Two main algorithms are discussed in this section. The first algorithm is found on the monotonic convergence behavior of deterministic ARE sequences, jointly with the above given computational scheme to solve ARE. The second algorithm will be developed as a direct iterative procedure derived from Equation (16).

**Algorithm 2**: Solution of Equation (8) is obtained as K = lim$K_i$ where $K_i$> 0; i = 1,2,... is the unique solution of the quadratic equation:

$$A^T K_i + K_i A - K_I A - K_i A - k_i B K_i^{-1} B^T K_i + L_i = 0\text{...................................}(29)$$

Where, recursively:

$$L_i = L + \sigma F^T K_{i-1} F\text{................................ ....................}(30.1)$$

$$R_i^{-1} = (\mathrm{R} + \sigma G^T K_{i-1} G)^{-1}\text{.......................................................}(30.2)$$

and $K_0$ = 0. Moreover, for the i<sup>th</sup> main iteration step, the following imbedded iterative procedure is applied to determine the solution $K_i$of Equation (29). Inner iterations for every i = 1,2...

**begin**:

i) Compute auxiliary coefficient matrices:

$$D_i = \frac{1}{2}(L_i A L_i^{-1} - A^T\text{..................................................} \quad (31.1)$$

$$T_i = L_i(AL_i^{-1}AT + BR_i^{-1}BT)\text{................................ ...........}(31.2)$$

ii) Starting from the initial value:

$X_{i,0} = L_{i-i}K_{i-1}^{-1} + A^T$; $X_{0,0} = T_0$................................ ...........................(32)

Compute, recursively:

$$X_{i,j+1} = D_i + \frac{1}{2}(X_{i,j}X_{i,j}^{-1})\text{...................................................}(33)$$

Forj = 1,2,… and stop iterations for some j = N, $\|X_{i,\ N+1} - X_{i,\ N}\| < \varepsilon_0$ is the admissible error bound.

iii) Let $\widehat{X}$ = $X_{i}$, $_{N+1}$ be the last computed matrix in (ii) then by using this value, compute$K_i$ by formula:

$K_i$= ($\widehat{X}_i$ - $A^T$)$^{-1}$ $L_i$……………………………. ……………………….(34)

Outer iterations on the sequences $K_i$i = 0,1,2,… will use the stopping criteria ||KN+1 - KN||<$\varepsilon_{i,}$ for some i = N and some admissible tolerance $\varepsilon_{i.}$ Observe that in Equation (32), the last computed value $K_{i-1}$during the (i-1)$^{th}$outer iteration is used to obtain an initial guess for the i$^{th}$ inner iteration step. An experience with this algorithm shows like in theklainman's method, only one to four inner iterations are required to get $k_i$from $K_{i-1}$, for i >3.

**End**

**Algorithm 3**: Solution of Equation (8) is obtained as:

K = ($\widehat{X}$ – AT)$^{-1}$L……………………………..………………………………………………(35)

Where $\widehat{X}$is the solution of Equation (16), and is computed recursively as follows:
For each i = 0,1,2,3… and choosing $X_0$ = 0:

**Begin**

i) Compute the auxiliary matrix:

$T(X_i) = LAL^{-1}A^TLB[R + \rho G^T(x_i - A^T)^{-1}LG]^{-1}B^T - \sigma(x_i - A^T)F^T(z_i - A^T)^{-1}LFL^{-1}(x_i - A^T)$……………………………. …………………...(36)

ii) Iterate, by using recursive formula:

$X_{i+1} = D + \frac{1}{2}(x_i + T(x_i)x_i^{-1})$………………………………………… (37)

Then, do i = i+1 and go to step (i).

**End**

As in Algorithm 3, the stopping criteria will be ||$K_{N+1}$ - $K_N$||<$\varepsilon$, for some i = N and some given tolerance $\varepsilon$. A necessary condition for convergence of this algorithm is that T($x_i$)> 0, for all i=1,2,…. However, no monotones convergence properties can be given for this case. Computational experience with this procedure shows that it can be advantageous over Algorithm 2 when the more general noise levels are small (i.e. ||$\sigma$F|| and ||$\rho$G<<1). For more general problems Algorithm 2 is preferable since it assures monotonic convergence towards solution if the existence conditions are fulfilled (i.e., if the stochastic system is controllable and observable).

We stress the computational advantages of those algorithms mainly due to the fact that the only operations involved in implementation are matrix additions, products and inversions in the field of positive definite matrices, (see equations 30.1 - 34). Furthermore, the set of transformations applied on the former SARE in Equation (8) to get polynomial like form in Equation (16) enables us to obtain additional computational advantages when system matrix is a band diagonal matrix and the stochastic components:|| F || and || G || are small.

Under those conditions,$T^{\frac{1}{2}}$ with T as given by Equation (19), gives a good first approximation of the solution of Equation (16) and then algorithms 2 and 3 may converge in a few program rounds. This is the case of many distributed parameter systems in which the finite elements quantization state is functionally linked with a few neighbor state variables. Thus, this leads to the typical band diagonal structure of the system matrix, in order to improve convergence of the iteration in Equations (27), (33) and (37). Those equations can be substituted by the more general procedures such as:

$$X_{i+1} = -\frac{2}{\mu}D + \frac{1}{\mu}\left((\mu-1)X_i + TX_i^{-1}\right), \mu > 2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(38)}$$

and, on the computer implementation, μ is estimated by means of a special sub routine in order to ensure convergence and to minimize the number of iterations.

## 4. RESULTS AND DISCUSSION

### 4.1 OPERATIONS COUNT AND STORAGE

We shall give an approximate operation count for the solution of n<sup>th</sup> order SARE of the form in Equation (8) by utilizing Algorithm 2. Each operation is assumed to be equivalent to compute a+(b x c) where, b, c are floating-point numbers.

It is almost impossible to give an accurate operation count for the proposed algorithm mainly due to the fact that the number of inner and outer iterations strongly depends on problem structure, numerical values of coefficient matrices and noise covariance $\sigma$ and $\rho$. We shall give only a ballpark 0 ($n^3$) figure for the entire process. The main steps are

a) Outer iterations operation:
   i) Computation of $L_i$ and $R_i^{-1}N_0$ ($5n^3$)
b) Inner iterations operation:
   i) Computation of $D_i$; $T_i 7n^3$
   ii) Computation $\hat{x}_i N_i$ ($7/3n^3$)
   iii) Computation of $K_i$($7/3n^3$)

$N_i$ represents the number of steps required to compute $K_i$from $K_{i-1}$.
For a given number of figures suing iteration procedure (31.1-34); $N_0$ represents the total number of outer iterations required to obtain K. Thus, we have a ballpark estimate of about ($40 + 7N_i)/3n^3$ $N_0$ operations for the entire process. By solvinga large number of test problems it can be estimated that the total number of inner iterations tends to be approximately equal to three times the total number of outer iterations, which gives an average estimate of $\overline{N}_i = 3$.

Comparison with other methods, like the kleinman's method and the Schur method (Laub, 2009) can now be made on the basis of this average number of inner iterations which is shared by most methods. By utilizing the most today efficient Hessenberg-Schur method (Geolude *et al*., 2009) the solution of one linear matrix Liapunov equation requires about $20n^3$ operations: Hence, the total cost by using Kleinman's method to solve SARE is by $65n^3$ $N_0$ operations.

By utilizing the well-known and efficient Schur method to solve deterministic ARE the computational cost is about $75n^3$ operations per equation. Thus, to solve SARE by using this inner procedure the total computation cost is roughly given $75n^3$ $N_0$ operations. Finally, by utilizing Algorithm 2 given in this paper the total computational cost are about $20n^3N_0$ operations. Thus, we see that the cost proposed method significantly reduces the computational cost for a large class of SARE problems, as compared to the most efficient available algorithms today.

With respect to storage considerations, the algorithm requires 6n$^2$ storage locations, less than a half of the required storage by using the "eigenvector methods" (Larin*et al*., Laub, 2009) which are limited to solving matrix equations on the order of 100 or less in many common computing environments. Additionally, implementation of the proposed algorithms in this paper only requires standard routines for matrix inversion, addition and product which are immediately available in all computer libraries.

Application of Algorithm 3 must be restricted to SARE computations for which $\|\sigma F\|$ and $\|\rho G\|$ are small to assure convergence. When these conditions are satisfied the computational results presented in Incerttis (2002) enable us to determine approximate bounds for operations counts and required storage.

Computational experiences with both algorithms show effectiveness for low or moderate values of $\|\sigma F\|$ and $\|\rho G\|$. As the stochastic components increase computational burden also increases, numerical stability decreases and the solution norm $\|K\|$tends to be unbounded for very high noise components. We should remark that the results in this paper can be generalizedto the case of several independent Noise $a_i$, with covariance $\sigma$i, and $\beta$j with covariancespj by replacing stomach components in Equation (8) by the general terms:

$$\textstyle\sum_i \sigma_i F_i^T \text{ K F}_i \qquad \text{and} \qquad \sum_i \rho_j G_j^T \text{ K F}$$

To end this section it is necessary to remark that the proposed approach can be extended to the case of a singular matrix L by means of perturbation theory or by an extension of the formulation given in Incertis(2002) for ARE problems.

## 5. WORKED EXAMPLE

The following example of a system with state dependent noise and zero control noise illustrates the usefulness of Algorithm 2 and the unstable behavior of the solutions for increasing values of $\sigma$. Let us consider a system of the form in Equations (1) to (6) where the system and performance matrices are given by:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; \ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; F = \begin{bmatrix} 0\ .5 & 0.1 \\ 0\ .2 & 0.5 \end{bmatrix}; C = I; R = I; = 1; G = 0$$

By utilizing Algorithm 3 to solve this problem equation adopts the particular form:

$$T\ (X_i) = T - \sigma P_i\ F^T\ P^{-1}{}_i\ LFL\ {}^{-1}P_i\ ;\ P_i = X_i - A^T \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(36)$$

Where,

$$T = LAL^{-1}\ A^T + LBR^{-1}\ B^T = I \text{ and } L = I \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(37)$$

Also, matrix D in (37) is given by:

$$D = \frac{1}{2}\ (LAL^{-1} - A^T) = \begin{bmatrix} 0 & 0.5 \\ -0.5 & 0 \end{bmatrix}$$

Table 1 shows the computational effort versus the covariance noise parameter $\alpha$ in Algorithm 3. The results show the effectiveness of Algorithm 2 for small or intermediate of $\sigma$. Increase toward values for which the stochastic system cannot be stabilized by feedback, the computational burden increases and the SARE solution norm grows proving that state-department noise calls for vigorous control (large gains).

**Table 1**
**Noise Covariance, Number of iterations, and Maximum Elements in K**

| Noise Covariance | Number of iterations | Max. Element in K |
|---|---|---|
| 0 | 10 | 1.73205 |
| 0.25 | 12 | 1.84523 |
| 0.5 | 13 | 1.96614 |
| 0.75 | 13 | 2.09412 |
| 1 | 14 | 2.22838 |
| 2 | 20 | 2.83521 |
| 3 | 35 | 3.54571 |
| 4 | 136 | 4.35715 |
| 5 | 500 | - |

## 6. CONCLUSION

We have discussed in considerable detail some new theoretical results and extensions leading to new algorithms for solving matrix equations arising in stochastic system analysis and optimizations.A number of numerical issues have been addressed and the detailed formulation of the algorithms has been given. The method performs reliably well on large dimensional systems with a band diagonal structure in the system matrix in practice than the actual available methods. It is more robust and saves a large amount of computer memory.

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