Dutse Journal of Pure and Applied Sciences (DUJOPAS) Vol. 4 No. 1 June 2018

ANALYSIS OF BACKWARD BIFURCATION IN SOME EPIDEMIOLOGICAL MODELS

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Abstract

In this paper, two epidemiological models were studied. It first investigated the mathematical model for tuberculosis transmission with heterogeneity in disease susceptibility and progression under a treatment regime for infectious cases. It then investigated the mathematical model for the assessment of the effect of traditional beliefs and custom on the transmission dynamics of the 2014 Ebola outbreak. At the end, it was shown that the first model exhibits the backward bifurcation phenomenon while the second model did not exhibit such characteristics. Furthermore, the parameter that caused the backward bifurcation was also identified.

Keywords: Bifurcation, Deterministic model, Dynamics, Transmission.

1.0 Introduction

In dynamical system, systems of physical interest typically have parameters which appear in the defining system of equations. As these parameters are varied, changes may occur in the qualitative structure of the solutions for certain parameter values. These changes are called bifurcations and the parameter values are called bifurcations values (Guckerheimer and Holmes, 1983).

Garba *et al.* (2008), considered a deterministic model for the transmission dynamics of a strain of dengue disease which allows transmission by exposed human and mosquitoes. The model had backward bifurcation phenomenon. It was also shown that when the model was extended by incorporating an imperfect vaccine against the strain of dengue, the backward bifurcation was further sustained. They found out that the standard incidence function was the cause of the backward bifurcation both in the basic and vaccination models.Sharomi and Gumel (2009) designed and analyzed a two group deterministic model for Chlamydia trachomatis was designed and analyzed to understand its transmission dynamics. The model had backward bifurcation. It was also shown that when the basic model was extended to incorporate the use of treatment for infectious individuals, backward bifurcation further was sustained.

Andrawus *et al.* (2017) investigates the causes of the backward bifurcation phenomenon in a mathematical model that described the dynamics of Tuberculosis-Dengue co-infection in a population where both diseases are endemic. This phenomenon is characterized by the coexistence of a stable disease-free equilibrium with a stable endemic equilibrium, when the associated reproduction number is less than unity. The analyses indicated that, for the Tuberculosis (TB)only model, exogenous reinfection and the reinfection of previously treated individuals are the causes of the backward bifurcation phenomenon, while for the Dengueonly model, disease-induced deaths in infected humans will lead to the backward bifurcation in the system. In co-infection scenarios where TB is having a larger disease burden than dengue, it is shown that the exogenous reinfection of latently infected TB individuals and the reinfection of previously treated individuals for TB will lead to the backward bifurcation phenomenon. The implication of these results is that for the reproduction numbers of the model to be useful for designing robust public health control measures against both diseases, concerted efforts must be geared towards minimizing the incidences of exogenous reinfection of latently infected TB cases, re-infection of previously treated individuals for TB and disease-induced deaths due to dengue infection.

Andrawus and Eguda (2017) considered a phenomenon of backward bifurcation in a syphilis model. It was found that the loss of transitory (natural) immunity after a successful treatment was the cause of backward bifurcation in the model. In the same vain, Villavicencio-Pulido *et al*. (2013), considered the existence of backward bifurcation in some infectious disease. They investigated how the backward bifurcation can be generated and preserved or eliminated. Anguelov et al. (2014) presented a two stage SIS epidemiological model in animal population with bovine tuberculosis (BTB) in African buffalo as a guiding example. The analysis done in the proposed model revealed that the model exhibits the backward bifurcation phenomenon and is caused by imperfect vaccine.

Hussaini *et al*. (2016) design a new model for the transmission dynamics of visceralleishmani- asis(AVL) and human immune deficiency virus (HIV) in a community and used to Assess the impact of the co-endemicity of the two diseases on the dynamics of each disease. The AVL-only component of the model undergoes the phenomenon of backward bifurcation when the AVL-associated mortality is non zero.

This Paper is aimed at determining the causes of backward bifurcation in two epidemiological models namely, (Okuonghae, 2013) and (Augusto *et al.*, 2015) in which backward bifurcation were not treated.

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2.1 Model Analysis to Determine Backward Bifurcation

2.2 Model I

Consider the model of tuberculosis transmission with heterogeneity in disease susceptibility and progression (Okuonghae, 2013).

$$\dot{S}_1(t) = h_1 \wedge - \lambda S_1 - \mu S_1 \tag{2.1}$$

$$\dot{S}_2(t) = h_2 \wedge - x\lambda S_2 - \mu S_2 \tag{2.2}$$

$$\dot{S}_3(t) = h_3 \wedge - \mu S_1 \tag{2.3}$$

$$\dot{E}_1(t) = (1 - p_1)[\lambda(f_{11}S_1 + f_{21}xS_2 + g_{11}zT_1 + g_{21}yzT_2)] - \varepsilon\lambda E_1 - a_{11}E_1 \tag{2.4}$$

$$\dot{E}_2(t) = (1 - p_2)[\lambda(f_{12}S_1 + f_{22}xS_2 + g_{12}zT_1 + g_{22}yzT_2)] - \varepsilon x_3\lambda E_2 - a_{22}E_2 \tag{2.5}$$

$$\dot{E}_3(t) = \lambda(f_{13}S_1 + f_{23}xS_2 + g_{13}zT_1 + g_{22}yzT_2) - a_{33}E_3 \tag{2.6}$$

$$\dot{I}_1(t) = p_1\lambda[f_{11}S_1 + f_{21}xS_2 + g_{11}zT_1 + g_{21}yzT_1] + (\varepsilon\lambda + k_1)E_1 - a_{44}I_1 \tag{2.7}$$

$$\dot{I}_2(t) = p_2\lambda[f_{12}S_1 + f_{22}xS_2 + g_{12}zT_1 + g_{22}yzT_2] + (\varepsilon\lambda x_3 + k_2)E_2 + k_3E_3 - a_{55}T_2 \tag{2.8}$$

$$\dot{T}_1(t) = m_1r_1I_1 - zT_1\lambda - \mu T_1 \tag{2.9}$$

$$\dot{T}_2(t) = m_2r_2I_2 - yzT_2\lambda - \mu T_2 \tag{2.10}$$

$$\dot{R}(t) = a_{66}I_1 + a_{77}I_2 - \mu R \tag{2.11}$$

where,

$$a_{11} = k_1 + \mu, \quad a_{22} - k_2 + \mu, \quad a_{33} = k_3 + \mu \; a_{44} = r_1 + \mu + d, + n, \quad a_{55} = r_2 + \mu + d_2 + n_2$$

$$a_{66} = (1 - m_1)r_1 + n_1, \quad a_{77} = (1 - m_2)r_2 + n_2, \; \lambda = \frac{\beta(bI_1 + I_2)}{N} \tag{2.12}$$

$$N = S_1 + S_2 + S_3 + E_1 + E_2 + E_3 + I_1 + I_2 + T_1 + T_2 + R \tag{2.13}$$

Table 2.1. Description of the variables and parameters used in model I

Variable	Description
S_1	Susceptible with no resistance
S_2	Susceptible with partial resistance
S_3	Susceptible with complete resistance
E_1	Latently infected individual that are rapid progressionsto active TB
E_2	Latently infected individual that are normal progressions to active TB

E_3	Latently infected individuals that are no or very slow progressions to active TB
I_1	Infectious individuals from the class of rapid progressions
I_2	Infectious individuals from the normal progression as well as ery slow (if any) progression i.e. E_2 and E_3.
T_1	Effectively treated individual whose infections were generated by the rapid progressions
T_2	Effectively treated individual whose infections were generated by the normal or very slow (if any) progressions
R	Individuals who become latent (which we shall refer to as secondary latency) after failed treatment and those who became latent due to self-cure from the infectious classes I_1 and I_2
N	Total number of individuals involved in the dynamics
f_{ij}	Rate at which the newly infected from S_i – individuals *(i = 1,2)* join the three latently infected class (with subscript *i.j = 1,2,3*) with respective proportion f_{ij}
g_{ij}	Rate at which the new infected who come from T_i*(i = 1,2)* join the three latently infected class (with subscript *i.j = 1,2,3*) with respective proportion g_{ij}
P_i	Fraction of direct progressions to active tuberculosis from new infections *(i= 1,2)*.
μ	Natural death rate
Λ	Recruitment rate
β*,β	Transmission rate
k_i *i = 1,2,3*	Activation rate
n_1, n_2	Self – cure
m_1, m_2	Treatment success
r_1, r_2	Recovery rate
d_1, d_2	TB – induced death
h_1, h_2 h_3	Genome frequency

2.3 Disease Free Equilibrium (DFE) of Model I

The disease free equilibrium of model I can be obtained by setting the right hand side of the model to zero. This gives

$$S_1^0 = h_1 \frac{\Lambda}{\mu},\ S_2^0 = h_2 \frac{\Lambda}{\mu}\ \ S_3^0 = h_3 \frac{\Lambda}{\mu},\ E_1^0 = E_2^0 = E_3^0 + I_1^0 = I_2^0 = 0 = T_1^0 = T_2^0 = R^0 = 0$$

Hence, the DFE

$$\varepsilon_0 = \left(S_1^0, S_2^0, S_3^0, E_1^0, E_2^0, E_3^0, I_1^0, I_2^0, T_1^0, T_2^0, R^0\right) = \left(h_1 \frac{\Lambda}{\mu}, h_2 \frac{\Lambda}{\mu}, h_3 \frac{\Lambda}{\mu}, 0,0,0,0,0,0,0,0,0\right) \quad (2.14)$$

The associated reproduction number (Okuonghae 2013) is given by

$$R\tau = \beta\left[\frac{bk_1\tau_1}{a_{11}a_{44}}(1-p_1) + \frac{b\tau_1}{a_{44}} p_1 + \frac{k_2\tau_2}{a_{22}a_{55}}(1-p_2) + \frac{\tau_2}{a_{33}} p_2 + \frac{k_3\tau_3}{a_{33}a_{55}}\right]\frac{\omega}{\psi + \xi + \varsigma + \zeta + \varpi} \quad (2.15)$$

where,

$\omega = a_{11}a_{22}a_{33}a_{44}a_{55}$, $\psi = a_{22}a_{33}a_{55}bk_1\tau_1(1-p_1)$, $\xi = a_{11}a_{22}a_{33}a_{55}b\tau_1 p_1$,

$\varsigma = a_{11}a_{33}a_{44}k_2\tau_2(1-p_2)$

$\zeta = a_{11}a_{22}a_{44}a_{55}\tau_2 p_2$, $\varpi = a_{11}a_{22}a_{44}k_3\tau_3$

2.4 Determination of the backward bifurcation of Model I

Let $x_1 = S_1$, $x_2 = S_2$, $x_3 = S_3$, $x_4 = E_1$, $x_5 = E_2$, $x_6 = E_3$, $x_7 = I_1$, $x_8 = I_2$, $x_9 = T_1$, $x_{10} = T_2$, $x_{11} = R$

$$N = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} + x_{11}$$

The model (2.1) - (2.11) becomes.

$$\dot{x}_1 = h_1\Lambda - \frac{\beta(bx_7 + x_8)}{N}x_1 - \mu x_1 \tag{2.16}$$

$$\dot{x}_2 = h_2\Lambda - x\beta(bx_7 + x_8)x_2 - \mu x_2, \ \dot{x}_3 = h_3\Lambda - \mu x_3 \tag{2.17}$$

$$\dot{x}_4 = (1-p_1)f_{11}\frac{\beta(bx_7 + x_8)}{N}x_1 + (1-p_1)f_{21}\frac{x\beta(bx_7 + x_8)}{N}x_2 + (1-p_1)g_{11}\frac{z(bx_7 + x_8)}{N}x_9 + (1-p_1)g_{21}\frac{yz\beta(bx_7 + x_8)}{N}x_{10} - \frac{\varepsilon\beta(bx_7 + x_8)x_4}{N} - a_{11}x_4 \tag{2.18}$$

$$\dot{x}_5 = (1-p_2)f_{12}\frac{\beta(bx_7 + x_8)}{N}x_1 + (1-p_2)f_{22}\frac{x\beta(bx_7 + x_8)}{N}x_2 + (1-p_2)g_{12}\frac{z\beta(bx_7 + x_8)}{N}x_9 + (1-p_2)g_{22}\frac{yz\beta(bx_7 + x_8)}{N}x_{10} - \frac{\varepsilon\bar{x}\beta(bx_7 + x_8)}{N} - a_{22}x_5 \tag{2.19}$$

$$\dot{x}_6 = f_{13}\frac{\beta(bx_7 + x_8)}{N}x_1 + f_{23}\frac{x\beta(bx_7 + x_8)}{N}x_2 + g_{13}\frac{z\beta(bx_7 + x_8)}{N}x_9 + g_{23}\frac{yz\beta(bx_7 + x_8)}{N}x_{10} - a_{33}x_6 \tag{2.20}$$

$$\dot{x}_7 = p_1 f_{11}\beta(bx_7 + x_8)x_1 + p_1 f_{21}x\beta(bx_7 + x_8)x_2 + p_1 g_{11}z\beta(bx_7 + x_8)x_9 + p_1 g_{21}yz\beta(bx_7 + x_8)x_{10} + \varepsilon\beta(bx_7 + x_8)x_4 + k_1x_4 - a_{44}x_7 \tag{2.21}$$

$$\dot{x}_8 = p_2 f_{12}\frac{\beta(bx_7 + x_8)}{N}x_1 + p_2 f_{22}\frac{x\beta(bx_7 + x_8)}{N}x_2 + p_2 g_{12}\frac{z\beta(bx_7 + x_8)}{N}x_9 + p_2 g_{22}yz\beta(bx_7 + x_8)x_{10} + \varepsilon x_3\beta(bx_7 + x_8)x_5 + k_2x_5 + k_3x_6 - a_{55}x_8 \tag{2.22}$$

$$\dot{x}_9 = m_1r_1x_7 - \frac{z\beta(bx_7 + x_8)}{N}x_9 - \mu x_9, \tag{2.23}$$

$$\dot{x}_{10} = m_2r_2x_8 - \frac{yz\beta(bx_7 + x_8)}{N}x_{10} - \mu x_{10}, \tag{2.24}$$

$$\dot{x}_{11} = a_{66}x_7 + a_{77}x_8 - \mu x_{11} \tag{2.25}$$

The corresponding Jacobian $\boldsymbol{J_{\beta^*}}$ of the system (2.16)-(2.25) at the DFE is given as

$$
\begin{pmatrix}
-\mu & 0 & 0 & 0 & 0 & 0 & & -\beta\left(\frac{h_1}{h_1+h_2+h_3}\right) & 0 & 0 & 0 \\
0 & -\mu & 0 & 0 & 0 & 0 & -b\beta x\left(\frac{h_2}{h_1+h_2+h_3}\right) & -\beta x\left(\frac{h_2}{h_1+h_2+h_3}\right) & 0 & 0 & 0 \\
0 & 0 & -\mu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -a_{11} & 0 & 0 & -b\beta(1-p_1)\frac{(f_{11}h_1+xf_{21}h_2)}{h_1+h_2+h_3} & \beta(1-p_1)\frac{(f_{11}h_1+xf_{21}h_2)}{h_1+h_2+h_3} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -a_{22} & 0 & -b\beta(1-p_1)\frac{(f_{12}h_1+xf_{22}h_2)}{h_1+h_2+h_3} & \beta\frac{(f_{13}h_1+xf_{23}h_2)}{h_1+h_2+h_3} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -a_{33} & -b\beta\frac{(f_{13}h_1+xf_{23}h_2)}{h_1+h_2+h_3} & \beta\frac{(f_{13}h_1+xf_{23}h_2)}{h_1+h_2+h_3} & 0 & 0 & 0 \\
0 & 0 & 0 & K_1 & 0 & 0 & -a_{44}+b\beta p_1\frac{(f_{11}h_1+xf_{21}h_2)}{h_1+h_2+h_3} & \beta\, p_1\frac{(f_{11}h_1+xf_{21}h_2)}{h_1+h_2+h_3} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & K_2 & K_3 & b\beta p_2\frac{(f_{12}h_1+xf_{22}h_2)}{h_1+h_2+h_3} & -a_{55}+\frac{\beta p_2(f_{12}h_1+xf_{22}h_2)}{h_1+h_2+h_3} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & m_1r_1 & 0 & -\mu & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & m_2\,r_2 & 0 & -\mu & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & a_{66} & a_{77} & 0 & 0 & -\mu
\end{pmatrix}
$$

Suppose $\beta=\beta^*$ is chosen as the bifurcation parameter. At R_0=1,

$$\beta^*=\frac{a_{11}a_{22}a_{33}a_{44}a_{55}}{a_{22}a_{33}a_{55}bk_1\tau_1(1-p_1)+a_{11}a_{22}a_{33}a_{55}b\tau_1p_1+a_{11}a_{33}a_{44}k_2\tau_2(1-p_2)+a_{11}a_{22}a_{44}a_{55}\tau_2p_2+a_{11}a_{22}a_{44}k_3\tau_3}$$

The Jacobian $\boldsymbol{J_{\beta^*}}$ of (2.16)-(2.25)evaluated at the DFE when $\beta=\beta^*$ has a right eigenvector (corresponding to the zero eigenvalue) which is given as

$$w=[w_1,w_2,w_3,w_4,w_5,w_6,w_7,w_8,w_9,w_{10},w_{11}]^T$$

The left eigenvector (corresponding to the zero eigenvalue) is given as

$$v=[v_1,v_2,v_3,v_4,v_5,v_6,v_7,v_8,v_9,v_{10},v_{11}]$$

2.4.1 Components of w and v

The components of w and v are

$$w_1 = \frac{-\beta x_1^0}{\mu N^*}(bw_7 + w_8), w_2 = \frac{-\beta x_2^0}{\mu N^*}(bw_7 + w_8), w_3 = w_4 = 0,$$

$$w_5 = \left[\frac{(1-p)\beta b g_2}{N^*} w_7 + \frac{(1-p_2)\beta g_2}{N^*} w_8\right]\frac{1}{a_{22}}, \quad w_6 = \left[\frac{\beta b g_3}{N^*} w_7 + \frac{B g_3}{N^*} w_8\right]\frac{1}{a_{33}} \tag{2.26}$$

$$w_7 = w_7 > 0,\ w_8 = w_8 > 0,\ w_9 = \frac{m_1 r_1}{\mu} w_7, \quad w_{10} = \frac{m_2 r_2}{\mu} w_8,\ w_{11} = \frac{a_{66} w_7 + a_{77} w_8}{\mu}.$$

$$v_1 = v_2 = v_3 = v_9 = v_{10} = v_{11} = 0,\ v_4 = \frac{k_1}{a_{11}} v_7,\ v_5 = \frac{k_2}{a_{22}} v_8,\ v_6 = \frac{k_3}{a_{33}} v_8,\ v_7 = v_7 > 0,$$

$$v_8 = v_8 > 0. \tag{2.27}$$

$$N^* = x_1^0 + x_2^0 + x_3^0 = \frac{\Lambda}{\mu}(h_1 + h_2 + h_3) = \frac{\Lambda}{\mu}$$

$$g_1 = f_{11}x_1^0 + xf_{21}x_2^0,\ g_2 = f_{12}x_1^0 + xf_{22}x_2^0,\ g_3 = f_{13}x_1^0 + xf_{23}x_2^0$$

2.4.2 Formulation of a and b.

For the system (2.16)-(2.25) the associated non- zero second order partial derivatives at (DFE) were computed and bifurcation parameters are presented as follows:

$$a = \sum_{xij-1}^{11} v_k w_i w_j \frac{\partial f_k}{\partial x_i \partial x_j}(0,0)$$

$$a = 2v_4 A_3 w_9\left(w_7 + \frac{1}{b}w_8\right) + 2v_4 A_4 w_{10}\left(w_7 + \frac{1}{b}w_8\right) + 2v_5 A_7 w_9\left(w_7 + \frac{1}{b}w_8\right)$$

$$+ 2v_5 A_8 w_{10}\left(w_7 + \frac{1}{b}w_8\right) + 2v_6 A_{11} w_9\left(w_7 + \frac{1}{b}w_8\right) + 2v_6 A_{12} w_{10}\left(w_7 + \frac{1}{b}w_8\right)$$

$$+ 2v_7 A_{15} w_9\left(w_7 + \frac{1}{b}w_8\right) + 2v_7 A_{16} w_{10}\left(w_7 + \frac{1}{b}w_8\right) + 2v_8 A_{19} w_9\left(w_7 + \frac{1}{b}w_8\right) + 2v_8 A_{20} w_{10}\left(w_7 + \frac{1}{b}w_8\right)$$

$$+ 2v_4 A_1\left[w_1\left(w_7 + \frac{1}{b}w_8\right) - h_1 T^*\left(w_7 + \frac{1}{b}w_8\right)\right] + 2v_4 A_2\left[w_2\left(w_2 w_7 + \frac{1}{b}w_8\right) - h_2 T^*\left(w_7 + \frac{1}{b}w_8\right)\right]$$

$$+ 2v_5 A_5\left[w_1\left(w_7 + \frac{1}{b}w_8\right) - h_1 T^*\left(w_7 + \frac{1}{b}w_8\right)\right] + 2v_5 A_{10}\left[w_2\left(w_2 w_7 + \frac{1}{b}w_8\right) - h_2 T^*\left(w_7 + \frac{1}{b}w_8\right)\right]$$

$$+2v_6A_9\left[w_1\left(w_7+\frac{1}{b}w_8\right)-h_1T^*\left(w_7+\frac{1}{b}w_8\right)\right]+2v_6A_{10}\left[w_2\left(w_2w_7+\frac{1}{b}w_8\right)-h_2T^*\left(w_7+\frac{1}{b}w_8\right)\right]$$

$$+2v_7A_{13}\left[w_1\left(w_7+\frac{1}{b}w_8\right)-h_1T^*\left(w_7+\frac{1}{b}w_8\right)\right]+2v_7A_{14}\left[w_2\left(w_2w_7+\frac{1}{b}w_8\right)-h_2T^*\left(w_7+\frac{1}{b}w_8\right)\right]$$

$$+2v_8A_{17}\left[w_1\left(w_7+\frac{1}{b}w_8\right)-h_1T^*\left(w_7+\frac{1}{b}w_8\right)\right]+2v_8A_{18}\left[w_2\left(w_2w_7+\frac{1}{b}w_8\right)-h_2T^*\left(w_7+\frac{1}{b}w_8\right)\right]$$

(2.28)

where $T^*=\sum_{i=1}^{11} w_i$

$$A_1=\frac{\mu}{\Lambda}(1-p_1)\beta^*bf_{11},\quad \frac{A_1}{b}=\frac{\mu}{\Lambda}(1-p_1)\beta^*f_{11},\quad A_2=\frac{\mu}{\Lambda}(1-p_1)x\beta^*bf_{21},\quad \frac{A_2}{b}=\frac{\mu}{\Lambda}(1-p_1)x\beta^*bf_{21}$$

$$A_3=\frac{\mu}{\Lambda}(1-p_1)z\beta^*bg_{11},\quad \frac{A_3}{b}=\frac{\mu}{\Lambda}(1-p_1)z\beta g_{11},\quad A_4=\frac{\mu}{\Lambda}(1-p_1)zy\beta^*bf_{21};\quad \frac{A_4}{2}=\frac{\mu}{\Lambda}(1-p_1)zy\beta^*g_{21}$$

$$A_5=\frac{\mu}{\Lambda}(1-p_2)\beta^*bf_{12};\quad \frac{A_5}{b}=\frac{\mu}{\Lambda}(1-p_2)\beta^*f_{12};\quad A_6=\frac{\mu}{\Lambda}(1-p_2)x\beta^*bf_{22};\quad \frac{A_6}{2}=\frac{\mu}{\Lambda}(1-p_2)x\beta^*f_{22}$$

$$A_7=\frac{\mu}{\Lambda}(1-p_2)z\beta^*bg_{12};\quad \frac{A_7}{b}=\frac{\mu}{\Lambda}(1-p_2)z\beta^*g_{12};\quad A_8=\frac{\mu}{\Lambda}(1-p_2)yz\beta^*bg_{22};\quad \frac{A_8}{b}=\frac{\mu}{\Lambda}(1-p_2)yz\beta^*g_{22}$$

$$A_9=\frac{\mu}{\Lambda}\beta^*bf_{13};\qquad \frac{A_9}{b}=\frac{\mu}{\Lambda}\beta^*f_{13};$$

$$A_{10}=\frac{\mu}{\Lambda}x\beta^*bf_{23};\quad \frac{A_{10}}{b}=\frac{\mu}{\Lambda}\beta^*xf_{23};\quad A_{11}=\frac{\mu}{\Lambda}\beta bzg_{13};\quad \frac{A_{11}}{b}=\frac{\mu}{\Lambda}\beta zg_{13}$$

$$A_{12}=\frac{\mu}{\Lambda}yz\beta bg_{23};\quad \frac{A_{12}}{b}=\frac{\mu}{\Lambda}yz\beta^*g_{23};\quad A_{13}=\frac{\mu}{\Lambda}p_1\beta^*bf_{11};\quad \frac{A_{13}}{b}=\frac{\mu}{\Lambda}p_1\beta f_{11}$$

$$A_{14}=\frac{\mu}{\Lambda}p_1\beta bxf_{21};\quad \frac{A_{14}}{b}=\frac{\mu}{\Lambda}p_1\beta^*xf_{21};\quad A_{15}=\frac{\mu}{\Lambda}p_1\beta bzg_{11};\quad \frac{A_{15}}{b}=\frac{\mu}{\Lambda}p_1\beta^*zg_{11}$$

$$A_{16}=\frac{\mu}{\Lambda}p_1\beta^*byg_{21};\quad \frac{A_{16}}{b}=\frac{\mu}{\Lambda}p_1\beta yzg_{21};\quad A_{17}=\frac{\mu}{\Lambda}p_2\beta bf_{12};\quad \frac{A_{17}}{b}=\frac{\mu}{\Lambda}p_2\beta g_{12}$$

$$A_{18}=\frac{\mu}{\Lambda}p_2\beta^*bxf_{22};\quad \frac{A_{18}}{b}=\frac{\mu}{\Lambda}p_2\beta^*xf_{22};\quad A_{19}=\frac{\mu}{\Lambda}p_2\beta^*bzg_{12};\quad \frac{A_{19}}{b}=\frac{\mu}{\Lambda}p_2\beta^*zg_{12}$$

(2.29)

$$A_{20}=\frac{\mu}{\Lambda}p_2\beta byzg_{22};\qquad \frac{A_{20}}{b}=\frac{\mu}{\Lambda}p_2\beta yzg_{22}$$

Substitute for the A's

$$2\frac{\mu}{\Lambda}\beta^*(bw_7+w_8)[w_9(v_4(1-p_1)g_{11}+v_5(1-p_2)g_{12}+v_6g_{13}+v_7p_1g_{11}+v_8p_2g_{12})+w_{10}(v_4(1-p_1)yg_{21}$$

$$+v_5(1-p_2)yg_{22}+v_6yg_{23}+v_7p_1g_{21}y+v_8p_2yg_{22})]z$$

$$-2\frac{\mu}{\wedge}\beta^*(bw_7+w_8)\left[(w_1+h_1T^*)(v_4(1-p_1)f_{11}+v_5(1-p_2)fg_{12}+v_6f_{13}+v_7p_1f_{11}+v_8p_2f_{12})\right.$$

$$\left.+(w_2+h_2T)(v_4(1-p_1)xf_{21}+v_5(1-p_2)xf_{22}+v_6xf_{23}+v_7x_{23}+v_7p_1xf_{21}+v_8p_2xf_{22})\right]$$

$$2\frac{\mu}{\wedge}\beta^*\left(bw_7+\frac{w_8}{b}\right)[m_{11}z-m_{12}] \tag{2.30}$$

where

$$m_{11}=[w_9(v_4(1-p_1)g_{11}+v_5(1-p_2)g_{12}+v_6g_{13}+v_7p_1g_{11}+v_8p_2g_{12})+w_{10}(v_4(1-p_1)yg_{21}$$
$$+v_5(1-p_2)yg_{22}+v_6yg_{23}+v_7p_1g_{21}y+v_8p_2yg_{22})]$$
$$m_{12}=[(w_1+h_1T^*)(v_4(1-p_1)f_{11}+v_5(1-p_2)fg_{12}+v_6f_{13}+v_7p_1f_{11}+v_8p_2f_{12})$$
$$+v_5(1-p_2)yg_{22}+v_6yg_{23}+v_7p_1g_{21}y+v_8p_2yg_{22})] \tag{2.31}$$

Hence a > 0 if and only if $z>\frac{m_{11}}{m_{12}}$

$$b=v_k\sum_{i=1}^{7}w_i\frac{\partial^2 f_k(0,0)}{\partial x_i\partial\beta}$$
$$=v_4[w_7b(U_1+U_2)+w_8(U_1+U_2)]+v_5[w_7b(U_3+U_4)+w_8(U_3+U_4)]$$
$$+v_6[w_7b(U_5+U_6)+w_8(U_5+U_6)]+V_7[w_7b(U_7+U_8)+w_8(U_7+U_8)]$$
$$+v_8[w_7(U_9+U_{10})b+w_8(U_9+U_{10})] \tag{2.32}$$

where $U_1=\frac{f_{11}(1-p_1)x_1^0}{N*}$, $U_2=\frac{xf_{21}(1-p_1)x_2^0}{N*}$, $U_3=\frac{f_{12}(1-p_2)x_1^0}{N*}$

$U_4=\frac{xf_{22}(1-p_2)x_2^0}{N*}, U_5=\frac{f_{12}x_1^0}{N*}, U_6=\frac{xf_{23}x_2^0}{N*}$

$U_7=\frac{f_{11}p_1x_1^0}{N*}$, $U_8=\frac{xf_{21}p_1x_2^0}{N*}$, $U_9=\frac{f_{12}p_2x_1^0}{N*}$, $U_{10}=\frac{f_{12}p_2x_1^0}{N*}$

$$b=[v_4(U_1+U_2)+v_5(U_3+U_4)+v_6(U_5+U_6)+v_7(U_7+U_8)+v_8(U_9+U_{10})]\phi>0$$

where $\phi=(bw_7+w_8)$.

Thus, b>0.

Since b>0,it is clear that model I exhibits a backward bifurcation at $R_0=0$.This is caused by the exogenous re-infection.

2.4 Model II

Now, consider the model for the transmission dynamics of 2014 Ebola outbreak (Augusto *et al.*, 2015).

$$\dot{S}_C(t) = \Pi_C - \lambda_C(I_{CE}, I_{CN}, D_C)S_C(t) - \mu_H S_C(t) \tag{2.33}$$

$$\dot{E}_C(t) = \lambda_C(I_{CE}, I_{CN}, D_C)S_C(t) - (\sigma_C + \mu_H)E_C(t) \tag{2.34}$$

$$\dot{I}_{CE}(t) = \sigma_C E_C(t) - (\alpha_C + \mu_H)I_{CE}(t) \tag{2.35}$$

$$\dot{I}_{CN}(t) = \alpha_C I_{CE}(t) - (\gamma_C + \mu_H)I_{CN}(t) \tag{2.36}$$

$$\dot{R}_C(t) = h\gamma_C I_{CN}(t) - \mu_H R_C(t) \tag{2.37}$$

$$D_C(t) = (1-h)\gamma_C I_{CN}(t) - \delta_C D_C(t) \tag{2.38}$$

where, $\lambda_C(I_{CE}, I_{CN}, D_C) = \dfrac{B_C\phi_C(I_{CE} + I_{CN} + I_C D_C)}{S_C + E_C + I_{CE} + I_{CN} + R_C + D_C}$ is the infection rate of the disease (in the community).

The associated variables and parameters of model II are described below:

$S_C(t)$	=	Population of susceptible in the community
$E_C(t)$	=	Population of exposed in the community
$I_{CE}(t)$	=	Population of symptomatic in the early state of Ebola virus infection
$I_{CN}(t)$	=	Population of non-hospitalized symptomatic individual
$R_C(t)$	=	Population of recovered individuals in the community
$D_C(t)$	=	Population of Ebola – deceased individuals in the community
Π_C	=	Recruitment rate
μ_H	=	Natural death rate
σ_C	=	Progression rate of symptomatic individuals in the community
α_C	=	Progression rate of early symptomatic individuals in the community
Υ_C	=	Recovery rates of symptomatic individuals in the community
h	=	fraction of symptomatic non-hospitalized individual in the community
δ_C	=	cremated rate of Ebola-deceased individuals in the community.
τ_C	=	modification parameter.

2.5 Disease Free Equilibrium of Model II

The disease free equilibrium (DFE)of model II can be obtained by setting the right hand side of the model to zero.This result in the following

$$S_C^0 = \frac{\Pi_C}{\mu_H},\ E_C^0 = 0,\ I_{CE}^0 = 0,\ I_{CN}^o = 0,\ R_C^0 = 0,\ D_C^0 = 0$$

Hence model II has DFE given as

$$\varepsilon_0 = \left(S_C^0\ , E_C^0\ , I_{CE}^0 , I_{CN}^0 , R_C^0 , D_C^0\ \right) = \left(\frac{\Pi_C}{\mu_H}, 0,0,0,0,0\right)$$

The associated basic reproduction number (Augusto *et al.*, 2015) is given by

$$R_0 = \frac{B_C \phi_C \sigma_C}{k_1 k_2 k_3 \sigma_C}\left(\phi_C\left(\alpha_C + k_3\right) + T_C \alpha_C Y_C (1-h)\right) \tag{2.39}$$

where, $k_1 = \sigma_C + \mu_H$, $k_2 = \alpha_C + \mu_H$, $k_3 = Y_C + \mu_H$

2.6 Determination of backward bifurcation of model II

Let $x_1 = S_C, x_2 = E_C, x_3 = I_{CE}, x_4 = I_{CN}\ x_5 = R_C, x_6 = D_C$

The system is (2.33) - (2.38) becomes

$$f_1 = \dot{x}_1 = \Pi_C - \frac{B_C \phi_C \left(x_3 + x_4 + \tau_C x_6\right)}{x_1 + x_2 + x_3 + x_4 + x_5 + x_6} x_1 - \mu_H x_1 \tag{2.40}$$

$$f_2 = \dot{x}_2 = \Pi_C - \frac{B_C \phi_C \left(x_3 + x_4 + T_C x_6\right)}{x_1 + x_2 + x_3 + x_4 + x_5 + x_6} x_1 - \left(\sigma + \mu_H\right) x_2 \tag{2.41}$$

$$f_3 = \dot{x}_3 = \sigma_C x_2 - \left(\alpha_C + \mu_H\right) x_3, \tag{2.42}$$

$$f_4 = \dot{x}_4 = \alpha_C x_3 - \left(Y_C + \mu_H\right) x_5, \tag{2.43}$$

$$f_5 = \dot{x}_5 = h Y_C x_4 - \mu_H x_5 \tag{2.44}$$

$$f_6 = \dot{x}_6 = (1-h) Y_C x_4 - \delta_C x_6 \tag{2.45}$$

The Jacobian (J) of the system (2.40)- (2.45) at the DFE is given by

$$J = \begin{pmatrix} -\mu_H & 0 & -\beta_C^* \phi_C & -\beta_C^* \phi_C & 0 & -\beta_C^* \tau_C \phi_C \\ 0 & -\left(\sigma_C + \mu_H\right) & \beta_C^* \phi_C & \beta_C^* \phi_C & 0 & \beta_C^* \tau_C \phi_C \\ 0 & \sigma_C & -\left(\alpha_C + \mu_H\right) & 0 & 0 & 0 \\ 0 & 0 & \alpha_C & -\left(Y_C + \mu_H\right) & 0 & 0 \\ 0 & 0 & 0 & h Y_C & -\mu_H & 0 \\ 0 & 0 & 0 & (1-h) Y_C & 0 & -\delta_C \end{pmatrix}$$

Suppose $\beta_C = \beta_C^*$ is chosen as the bifurcation parameter.

At $R_0 = 1$, $\beta_C^* = \dfrac{k_1 k_2 k_3 \delta_C}{\phi_C \delta_C \left[\delta_C \left(\alpha_C + K_3\right) + \tau_C \alpha_C Y_C (1-h)\right]}$

2.6.1 Eigen vectors of J

The right and left eigenvectors of the Jacobian J at $\beta_C = \beta_C^*$ is computed in this section.

The right eigenvector (corresponding to the zero eigenvalue) is given by $w = \left[w_1, w_2, w_3, w_4, w_5, w_6\right]^T$. The left eigenvector is given as $v = \left[v_1, v_2, v_3, v_4, v_5, v_6\right]$.

The components of w and v are presented as follows:

$$w_1 = \frac{-\beta_C^*\phi_C w_3 - \beta_C^*\phi_C w_4 - \beta_C^*\phi_C \tau_C w_6}{\mu_H}, w_2 = w_2 > 0$$

$$w_3 = \frac{\sigma_C}{\alpha_C + \mu_H} w_2,\ w_4 = \frac{\alpha_C}{Y_C + \mu_H} w_3,\ w_5 = \frac{hY_C}{\mu_H} w_4,\ w_6 = \frac{(1-h)Y_C}{\delta_C} w_4 \tag{2.46}$$

$$v_1 = 0, v_2 = v_2 > 0, v_3 = \frac{\beta_C^*\phi_C v_2 + \alpha_C v_4 - \beta_C^*\phi_C v_1}{(\alpha_C + \mu_H)} \tag{2.47}$$

$$v_4 = \frac{\beta_C^*\phi_C v_2 + hY_C v_5 + (1-h)Y_C v_6 - \beta_C^*\phi_C v_1}{(Y_C + \mu_H)}, v_5 = 0, v_6 = \frac{\beta_C^*\tau_C\phi_C v_2 - \beta_C^*\tau_C\phi_C v_1}{\delta_C}$$

2.6.2 Computation of a and b

The associated non-zero partial derivatives of the system (2.40) - (2.45) were computed and the bifurcation parameters are found to be the following:

$$a = \sum_{kij=1}^{6} v_k w_i w_j \frac{\partial^2 f_k}{\partial x_i \partial x_j}(0,0) = v_2 \sum_{ij=1}^{6} w_i w_j \frac{\partial^2 f_2}{\partial x_i \partial x_j}$$

$$= -\frac{2\beta_C^*\phi_C\mu_H}{\Pi_C}(w_2w_3 + w_2w_4 + w_3w_5 + w_4w_5)v_2 - 2\beta_C^*\phi_C\left(\frac{\mu_H}{\Pi_C} + \frac{\mu_H}{\Pi_C}\tau_C\right)[w_3w_6 + w_4w_6]v_2$$

$$-2\beta_C^*\phi_C\frac{\mu_H}{\Pi_C}w_4^2 v_2 - 2\beta_C^*\phi_C\tau_C\frac{\mu_H}{\Pi_C}(w_5w_6 + w_2w_6 + w_6^2)v_2$$

$$= -2\beta_C^*\phi_C\frac{\mu_H}{\Pi_C}(w_2w_3 + w_2w_4 + w_3w_5 + w_4w_5 + w_4^2)v_2 - 2\beta_C^*\phi_C\tau_C\frac{\mu_H}{\Pi_C}[w_5w_6 + w_2w_6 + w_6^2]v_2$$

$$-2\beta_C^*\phi_C\left(\frac{\mu_H}{\Pi_C} + \frac{\mu_H}{\Pi_C}\tau_C\right)[w_3w_6 + w_4w_6]v_2$$

$$= -2\beta_C^*\phi_C\frac{\mu_H}{\Pi_C}\Big[w_2w_3 + w_2w_4 + w_3w_5 + w_4w_5 + w_4^2 + w_3w_6 + w_4w_6 + \tau_C$$

$$(w_5w_6 + w_2w_6 + w_6^2 + w_3w_6w_4w_6)\Big]v_2$$

Hence a < 0.

Computing for b

$$b = \sum_{ki=1} v_k w_i \frac{\partial^2 f_k}{\partial x_i \partial \phi}(0,0) = v_2(w_3 + w_4 - \tau_C w_6)\phi_C$$

Therefore, b > 0.

Since b > 0,itimplies that the system does not exhibit backward bifurcation.

3.0 Discussion of Results

In this work, it was discovered that Model I exhibit backward bifurcation when $a > 0$ and$b > 0$. It is identified that model I undergoes a backward bifurcation when the basic reproduction number, $R_0 = 1$.The exogenous re-infection of latently infected individuals was the cause of the phenomenon of backward bifurcation in model I.

Model II do not exhibits a phenomenon of backward bifurcation because the bifurcation parameter $a < 0$.It is necessary to note that when one of the backward bifurcation parameters is less than zero then the model will not undergo backward bifurcation.

4. Conclusion

A system undergoes backward bifurcation when it has multiple endemic equilibria and the effective reproduction number is less than one. In this case, the classical epidemiological requirement of having the reproduction number less than one is no longer sufficient for the effective control or elimination of the disease.

This study has identified some epidemiological parameters that can cause the backward bifurcation phenomenon. The work of Okuonghae (2013) and Augusto *et al*.(2015) were investigated using centre manifold theorem to depict the causes of backward bifurcation. In the case of Augusto *et al*. (2015), it has been shown that no backward bifurcation occurred. In Okuonghae (2013), the exogenous re-infection of latently infected individual was the cause of the backward bifurcation.

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