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Assessment of Numerical Performance of Some Runge-Kutta Methods and New Iteration Method on First Order Differential Problems

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Abstract

This research focuses on the assessment of the numerical performance of some Runge-Kutta methods and New Iteration Method "NIM" for solving first-order differential problems. The assessment is conducted through extensive numerical experiments and comparative analyses. Accuracy, efficiency, and stability are among the key factors considered in evaluating the performance of the methods. A range of first-order differential problems with diverse characteristics and complexity levels is employed to thoroughly examine the methods' capabilities and limitations. The numerical investigation that is defined in the study as well as the results that are stated in the Tables, demonstrates that all the approaches produce extremely accurate results. However, the "NIM" was shown to be the most effective of the three methods used in this study. Conclusively, the "NIM" should be employed to solve first-order nonlinear and linear ordinary differential equations in place of Runge-Kutta Fourth order method (RK4M) and Butcher Runge-Kutta Fifth order method (BRK5M). In addition, BRK5M is more applicable and efficient than RK4M when solving first order ordinary differential problems.

Keywords: Runge-Kutta Schemes, New iteration technique, First order differential problems, Numerical efficiency, Error analysis.

INTRODUCTION

The accurate and efficient numerical solution of differential equations is significant in various scientific and engineering fields. Runge-Kutta Methods (RKM) have been widely used and established as reliable numerical techniques for this purpose. However, advancements in numerical methods continue to emerge, leading to the development of new approaches. The objective of this research is to evaluate and compare the numerical performance of selected

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RKM methods, such as the classical fourth order RKM and Butcher Runge-Kutta (BRK) fifth order method, with the New Iteration Method "NIM". The "NIM" is based on a novel approach that combines variational principles and iterative refinement. Ordinary Differential equations (ODe) are represented by the relation in (1)

$$\frac{dr}{dx} = r'(x) = f(x, r) \tag{1}$$

with initial condition $r(x_0) = r_0$ and $x_{t+1} = x_t + l$, where l is the step length within $[c, d]$

The result at the point x_{t+1} obtained by $r(x_{t+1})$ for $x_{t+1} = x_t + l$ can be obtained using any efficient numerical scheme (Arora *et al*., 2020; Agbeboh *et al*., 2020).

The resolution of ODe is necessary in a variety of different contexts. The fundamental laws of physics, mechanics, electricity, and thermodynamics as well as the modeling of population growth and expansion are incorporated in these numerous situations (Al-Jawary *et al*., 2020; Rabiei *et al*., 2023). ODes can be found in a wide variety of applications across the fields of mathematics, social sciences, and natural sciences (Koroche, 2021). Most scholars have been interested in RKMs since the invention of digital computers, and a great number of researchers have contributed to recent developments of the theory and the development of expanded RKMs (Karthick *et al*., 2023; Batiha *et al*., 2023; Gebregiorgis and Gonfa, 2021).

Numerous studies have been conducted on how to solve ordinary differential equations. If the equation is linear, it is not a major problem because it can be resolved analytically. Unfortunately, most fascinating differential equations resulting from modelling real-world problems are non-linear, making analytical methods to solve them challenging. Numerical techniques have been developed and have successfully solved these ordinary differential equations. Additionally, a wide variety of computer programs have been developed to help users solve these equations. Given the significance of numerical methods in approximating solutions to differential equations, it is crucial to assess and compare the performance of different methods, including the established fourth-order and fifth-order Runge-Kutta methods, alongside the emerging new iteration method.

The three selected methods may be considered older numerical techniques, but they continue to be used for several reasons. They have a historical significance as foundational methods in numerical analysis and serve as benchmarks for comparison. These methods are reliable, well-understood, and widely implemented in libraries and software packages. Their utilization allows for comparative analysis and evaluation of newer methods. Additionally, practical considerations such as computational efficiency and compatibility with existing codes contribute to their continued usage. Therefore, despite being older methods, they are still relevant and provide a basis for assessing and selecting appropriate numerical approaches.

Many scholars have undertaken in-depth research involving a comparative analysis of various numerical methods employed to address ordinary differential problems. This exploration involves a comprehensive examination of the effectiveness and performance of these methods when applied to such problems. Senthilnathan *et al*. (2018) conducted a comparison study of the computational performance between Taylor methods and Euler methods on ODEs. Their findings revealed that Taylor's Method is more efficient and accurate when compared to Euler's Method. Various methods such as Taylor's series method, Euler's method, Runge Kutta first, second, third and fourth orders were utilized to compare their performance on Odes by Jamali (2019). Dhokrat (2021) compared methods of Euler, Improved

Euler, third and fourth order Runge-Kutta on solving initial valued problems and discovered that the fourth order RKM gave better results and converges quicker to the true solution. Until now, there has been a noticeable absence of research endeavors that have undertaken a comparative investigation into the performance of the fourth-order Runge-Kutta method and the New iterative method for addressing first-order ordinary differential equations.

This research study introduces novel contributions to the field of numerical analysis in solving first-order differential problems. The key areas of novelty include the assessment of the "NIM", a comparative evaluation of RKMs, consideration of multiple performance metrics, analysis of a wide range of differential problems, and contributions to the existing literature. By evaluating the performance of "NIM" alongside established RKMs and incorporating diverse problem scenarios, this study enhances the understanding and effectiveness of numerical approaches for solving first-order differential problems. Overall, these novel contributions contribute to advancing the field of numerical analysis and its practical applications. Insofar as ODes are frequently used as mathematical models in many fields of science, engineering, and economics. In this study, some numerical approaches to tackling common problems will be examined and compared.

METHODOLOGY

The Runge Kutta Fourth Order Method

This technique was invented by two German mathematicians in the 19th century to proffer approximate solutions to ordinary differential equations. The scheme is most conversant because it is efficient, accurate, steady and ease to program. This method is noteworthy by their order in the logic that they agree with Taylor's series solution up to terms of l^r where r is the order of the method (Arora *et al*., 2020). It does not require earlier computational of advanced derivatives of $r(x)$ as in Taylor's series method. The RK4M is broadly used for solving initial value problems for ordinary differential equation (Lee *et al*., 2020; Okeke *et al*., 2019). The general expression and the stages of RK4M can be expressed as follows;

$$r_{t+1} = r_t + \frac{l}{6}(k_1 + 2k_2 + 2k_3 + k_4), \quad \text{where} \quad \begin{aligned} k_1 &= f(x_t, r_t) \\ k_2 &= f\left(x_t + \frac{l}{2}, r_t + \frac{l}{2}k_1\right) \\ k_3 &= f\left(x_t + \frac{l}{2}, r_t + \frac{l}{2}k_2\right) \\ k_4 &= f(x_t + l, r_t + k_3 l) \end{aligned} \tag{2}$$

The algorithm for computation in solving ODes problem using the RK4M is presented below

RK4M Algorithm

Step 1: Express the function $f(x,r)$ in manner such that $f(x,r) \in [c,d]$.

Step 2: Provide the initial estimate for x_0 and r_0.

Step 3: Choose the desired step size $l = \frac{d-c}{m}$, where m is number of steps.

Step 4: Input c, d, x_0, r_0, M .

Step 5: for t from 1 to M do.

Compute k_1, k_2, k_3 *and* k_4 as denoted in the RK4M method.

Step 6: Set $x_{t+l} \to x_t$, then

Compute $r_{t+1} = r_t + \frac{l}{6}(k_1 + 2k_2 + 2k_3 + k_4)$.

Step 7: Output x_t *and* r_t

Step 8: End the process if $x_t \geq d$ *such that* $\|r_{t+1} - r_t\| < \varepsilon$

The Butcher Runge Kutta Fifth Order Method

Butcher's fifth-order method is an extension of the RK4 method that offers even higher accuracy. By incorporating additional stages and coefficients, the technique achieves a higher order of convergence, resulting in more precise approximations. This method has been widely investigated and has demonstrated superior performance in terms of accuracy in various problem scenarios. If we consider the first-order ODes in (1), then, the BRK5 technique is concerned on calculating r_{t+1} based on $x_{t+1} = x_t + l$ through the following formulas

$$r_{t+1} = r_t + \frac{l}{90}(7k_1 + 32k_3 + 12k_4 + 32k_5 + 7k_6) \tag{3}$$

where

$$\begin{aligned}
k_1 &= f(x_t, r_t) \\
k_2 &= f\left(x_t + \frac{l}{4}, r_t + \frac{l}{4}k_1\right) \\
k_3 &= f\left(x_t + \frac{l}{4}, r_t + \frac{1}{8}k_1 l + \frac{1}{8}k_2 l\right) \\
k_4 &= f\left(x_t + \frac{l}{2}, r_t - \frac{1}{2}k_2 l + \frac{1}{8}k_3 l\right) \\
k_5 &= f\left(x_t + \frac{3}{4}l, r_t + \frac{3}{16}k_1 l + \frac{9}{16}k_4 l\right) \\
k_6 &= f\left(x_t + l, r_t - \frac{3}{7}k_1 l + \frac{2}{7}k_2 l + \frac{12}{7}k_3 l - \frac{12}{7}k_4 l + \frac{8}{7}k_5 l\right) \\
t &= 0,1,2,\ldots\ldots\ldots\ldots
\end{aligned} \tag{4}$$

The algorithm for solving ODes problem using the BRK5 scheme is presented in the following algorithm

BRK5 Algorithm

Step 1: Express the function $f(x, r)$ in a manner such that $f(x, r) \in [c, d]$.

Step 2: Provide the initial estimate for x_0 and r_0.

Step 3: Choose the desired step size $l = \frac{d-c}{m}$, where m is number of steps.

Step 4: Input c, d, x_0, r_0, M .

Step 5: for t from 1 to M do.

Compute k_1, k_2, k_3, k_4, k_5 *and* k_6 as denoted in the BRK5 method.

Step 6: Set $x_{t+l} \rightarrow x_t$, then

Compute $r_{t+1} = r_t + \frac{l}{90}\left(7k_1 + 32k_3 + 12k_4 + 32k_5 + 7k_6\right)$.

Step 7: Output x_0 *and* r_0.

Step 8: End the process if $x_t \geq d \text{ such that } \left\| r_{t+1} - r_t \right\| < \varepsilon$.

The New Iteration Method "NIM"

According to Ghosh *et al*., (2021), the new iteration method general expression can be written as follows;

$$L\left[r(x)\right] + N\left[r(x)\right] + G(x) = 0 \tag{5}$$

where x denotes the independent variable, $r(x)$ describes the unknown function, $G(x)$ represents a given function. $L[\bullet] = \frac{d}{dx}[\bullet]$ Signifies the linear operator, $N[\bullet]$ denotes the nonlinear operator, the inverse operator $L^{-1}[\ ] = \int_0^x dt$ is the nonlinear operator for the differential equation described by (1) which is given by

$$r(x) = f(x) + \int_0^x N\left[r(t)\right]dt \tag{6}$$

A general function equation is presented as $r(x) = f(x) + N(x)$, and the nonlinear operator can be decomposed as

$$\left\{ N\left(\sum_{i=0}^{\infty} R_t(x)\right) = N(R_0) + \sum_{t=0}^{\infty} \left\{ N\left(\sum_{t=0}^{s} R_t\right) - N\left(\sum_{t=0}^{s-1} R_t\right)\right\}\right. \tag{7}$$

which is equivalent to

$$\left\{ N\left(\sum_{i=0}^{\infty} R_t(x)\right) = f(x) + \sum_{t=0}^{\infty} \left\{ N\left(\sum_{t=0}^{s} R_t\right) - N\left(\sum_{t=0}^{s-1} R_t\right)\right\}\right. \tag{8}$$

$$\begin{cases} R_0 = f, \\ R_1 = N\left(R_0\right), \\ R_{t+1} = N\left(R_0 + R_1 + \cdots + R_t\right) - N\left(R_0 + R_1 + \cdots + R_{t-1}\right) \end{cases} \tag{9}$$

Then,

$$\left\{\left(R_0 + R_1 + \cdots + R_{t+1}\right) = N\left(R_0 + R_1 + \cdots + R_t\right)\right. \tag{10}$$

And

$$\left\{N\left(\sum_{i=0}^{\infty} R_t\left(x\right)\right) = f\left(x\right) + \sum_{t=0}^{\infty}\left\{N\left(\sum_{t=0}^{s} R_t\right) - N\left(\sum_{t=0}^{s-1} R_t\right)\right\}\right. \tag{11}$$

The t−term approximate solution of the "NIM" is expressed by the relation in

$$\left\{r\left(x\right) = r_0 + r_1 + r_2 \cdots + r_{t-1}\right. \tag{12}$$

The algorithm for computation in solving initial value problems by the New iteration scheme is presented below

New Iteration Algorithm

Input: Read M (Number of iterations); L(r); N(r); and f.

Step 1: $r_{-1} = 0,\ r_0 = f$.

Step 2: $for\ \mathrm{t} = 0, t \le M, t++, do$.

Step 3: $A_t = f\left(r_t\right) - f\left(r_{t-1}\right);$

Step 4: $r_{t+1} = f + L(\mathrm{r}_t) + A_t;$

Step 5: $r = r_{t+1}$.

Output: r .

Numerical Experiments

We introduce some first order ordinary differential problems as numerical investigation, to verify the applicability and compare the efficiency of the three algorithms discussed in previous section. The computational results are obtained by MAPLE package. The numerical solutions and absolute errors are calculated, and the findings are illustrated graphically. The numerical problems for the investigation are listed as

Problem 1: Considering solving the ODe problem using RK4, BRK5 and "NIM"

$$\frac{dr}{dx} = x^2 + xr\ ,\ r\left(0\right) = 1$$

within $0 \le x \le 1,\ l = 0.1$ and analytical solution: $\sqrt{\frac{\pi}{2}}e^{\frac{x^2}{2}} erf\left(\frac{x}{\sqrt{2}} + e^{\frac{x^2}{2}} - x\right)$

Problem 2: Considering solving the ODe problem using RK4, BRK5 and "NIM"

$$r' = x^2 - r,\ r_0 = 1,\ x_0 = 0,\ l = 0.1$$

Exact answer: $r(x) = x^2 - 2x + 2 - e^{-x}$

Problem 3: Considering solving the ODe problem using RK4, BRK5 and "NIM"

$$r' = 3x^2 r,\ r_0 = 2,\ x_0 = 1,\ l = 0.1$$

True answer: $r(x) = 2e^{x^3}$

Table 1: RK4, BRK5, "NIM" and Analytical Solutions for Problem 1

X	Analytical Solution	RK4 Solution	BRK5 Solution	"NIM" Solution
0.0	1.0000000000	1.0000000000	1.0000000000	1.0000000000
0.1	1.0053465218	1.0053464802	1.0053465223	1.0053465218
0.2	1.0228894624	1.0228893798	1.0228894637	1.0228894624
0.3	1.0551919637	1.0551918407	1.0551919659	1.0551919637
0.4	1.1053189529	1.1053187896	1.1053189563	1.1053189529
0.5	1.1769749725	1.1769747667	1.1769749772	1.1769749725
0.6	1.2746789919	1.2746787363	1.2746789982	1.2746789919
0.7	1.4039883184	1.4039879953	1.4039883263	1.4039883184
0.8	1.5717877696	1.5717873427	1.5717877795	1.5717877696
0.9	1.7866658536	1.7866652528	1.7866658652	1.7866658536
1.0	2.0594074053	2.0594065035	2.0594074186	2.0594074053

Table 2: Comparison of Errors for Problem 1

X	RK4 Error	BRK5 Error	"NIM" Error
0.0	0.0000000000	0.0000000000	0.0000000000
0.1	4.16045×10^{-8}	5.68635×10^{-10}	0.0000000000
0.2	8.26715×10^{-8}	1.30435×10^{-9}	0.0000000000
0.3	1.23028×10^{-7}	2.22730×10^{-9}	0.0000000000
0.4	1.63324×10^{-7}	3.35449×10^{-9}	0.0000000000
0.5	2.05804×10^{-7}	4.69579×10^{-9}	0.0000000000
0.6	2.55623×10^{-7}	6.24756×10^{-9}	0.0000000000
0.7	3.23029×10^{-7}	7.98279×10^{-9}	0.0000000000
0.8	4.26941×10^{-7}	9.83601×10^{-9}	0.0000000000
0.9	6.00768×10^{-7}	1.16805×10^{-8}	0.0000000000
1.0	9.01815×10^{-7}	1.32944×10^{-8}	0.0000000000

Table 1 presents the result of RK4, BRK5, and "NIM" numerical solution with the analytical solution concerning problem 1. The error comparison between the three methods is presented in Table 2.

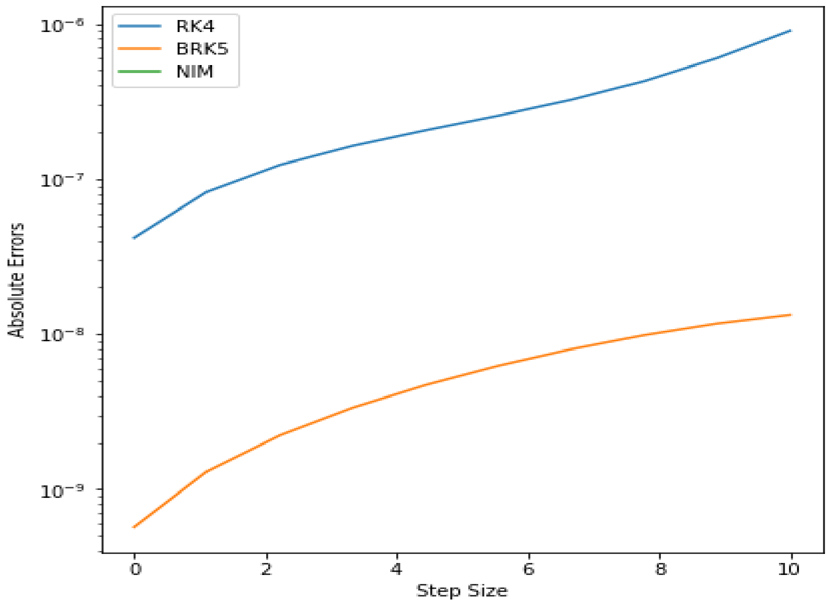

Figure 1: Error Plot for Problem 1

The plot in Figure 1 shows the computed errors of the three methods for problem 1.

Table 3: RK4, BRK5, "NIM" and Analytical Solutions for Problem 2

X	Analytical Solution	RK4 Solution	BRK5 Solution	"NIM" Solution
0.0	1.0000000000	1.0000000000	1.0000000000	1.0000000000
0.1	0.9051625819	0.9051627083	0.9051625828	0.9051625819
0.2	0.8212692469	0.8212694954	0.8212692486	0.8212692469
0.3	0.7491817793	0.7491821454	0.7491817817	0.7491817793
0.4	0.6896799539	0.6896804328	0.6896799571	0.6896799539
0.5	0.6434693402	0.6434699269	0.6434693441	0.6434693402
0.6	0.6111883639	0.6111890533	0.6111883683	0.6111883639
0.7	0.5934146962	0.5934154834	0.5934147011	0.5934146961
0.8	0.5906710358	0.5906719158	0.5906710413	0.5906710357
0.9	0.6034303402	0.6034313079	0.6034303462	0.6034303397
1.0	0.6321205588	0.6321216094	0.6321205652	0.6321205568

Table 4: Comparison of Errors for Problem 2

X	RK4 Error	BRK5 Error	"NIM" Error
0.0	0.0000000000	0.0000000000	0.0000000000
0.1	1.26369×10^{-7}	8.87522×10^{-10}	0.0000000000
0.2	2.48512×10^{-7}	1.70897×10^{-9}	0.0000000000
0.3	3.66090×10^{-7}	2.46888×10^{-9}	0.0000000000
0.4	4.78865×10^{-7}	3.17153×10^{-9}	0.0000000000
0.5	5.86686×10^{-7}	3.82094×10^{-9}	0.0000000000
0.6	6.89475×10^{-7}	4.42087×10^{-9}	0.0000000000
0.7	7.87213×10^{-7}	4.97486×10^{-9}	0.0000000000
0.8	8.79931×10^{-7}	5.48623×10^{-9}	2.0000×10^{-10}
0.9	9.67699×10^{-7}	5.95806×10^{-9}	6.0000×10^{-10}
1.0	1.05062×10^{-6}	6.39325×10^{-9}	1.9000×10^{-9}

Table 3 displays the "NIM", RK4, BRK5 and analytical solutions for problem 2, while the computed errors are presented in Table 4.

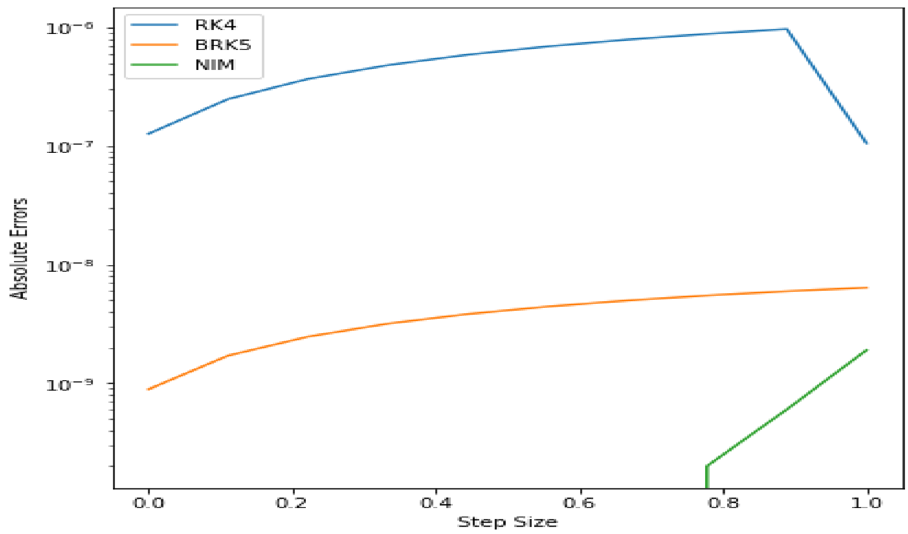

Figure 2: Error Plot for Problem 2

The above Figure shows the absolute errors between "NIM", RK4 and BRK5 for problem 2.

Table 5: RK4, BRK5, "NIM" and Analytical Solutions for Problem 3

X	Analytical Solution	RK4 Solution	BRK5 Solution	"NIM" Solution
0.0	2.0000000000	2.0000000000	2.0000000000	2.0000000000
0.1	2.0020010003	2.3652500000	2.3882824041	2.0020010003
0.2	2.0160641710	2.3706826250	2.3969069510	2.0160641710
0.3	2.0547356055	2.3886739902	2.4201690863	2.0547356055
0.4	2.1321847975	2.4266730232	2.4655302547	2.1321847975
0.5	2.2662969061	2.4923364073	2.5407305680	2.2662969061
0.6	2.4822047580	2.5937406124	2.6540088525	2.4822047580
0.7	2.8183375238	2.7396779435	2.8143902974	2.8183375238
0.8	3.3372502202	2.9400500531	3.0320467432	3.3372502202
0.9	4.1460131285	3.2063816548	3.3187390564	4.1460131284
1.0	5.4365636569	3.5524901118	3.6883563500	5.4365636523

Table 6: Comparison of Errors for Problem 3

X	RK4 Error	BRK5 Error	"NIM" Error
0.0	0.0000000000	0.0000000000	0.0000000000
0.1	0.3632489996	0.3862814038	0.0000000000
0.2	0.3546184539	0.3808427800	0.0000000000
0.3	0.3339383847	0.3654334808	0.0000000000
0.4	0.2944882256	0.3333454571	0.0000000000
0.5	0.2260395012	0.2744336618	0.0000000000
0.6	0.1115358544	0.1718040945	0.0000000000
0.7	0.0786595803	0.0039472264	0.0000000000
0.8	0.3972001671	0.3052034770	0.0000000000
0.9	0.9396314736	0.8272740720	0.0000000000
1.0	1.8840735450	1.7482073068	0.0000000000

Table 5 presents the result of RK4, BRK5, "NIM" and analytical solutions for problem 3. Their errors are displayed in Table 6.

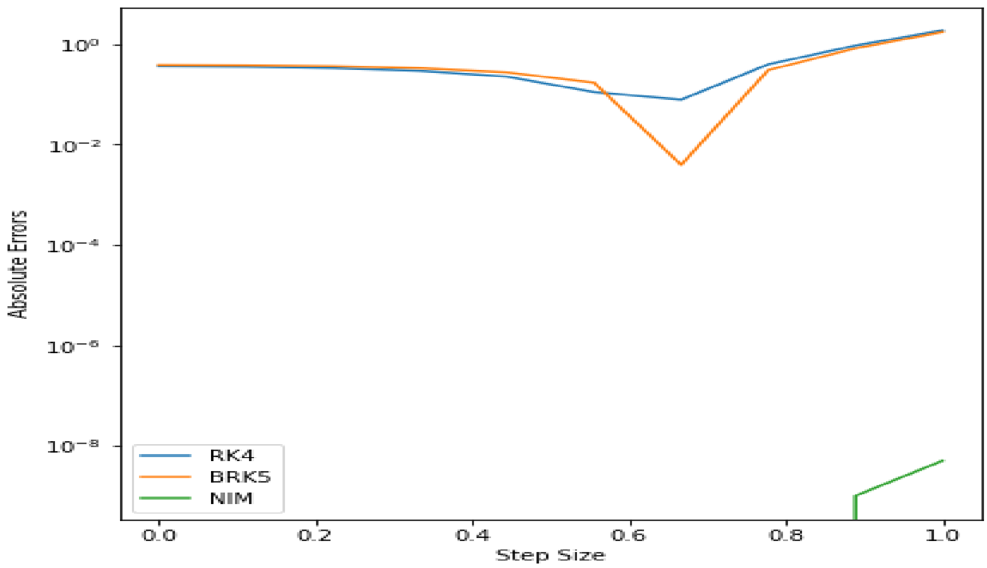

Figure 3: Error Plot for Problem 3

The plot in Figure 3 displays the graphical representation of computed error for new iteration method, RK4M and BRK5 for problem 3.

DISCUSSION

According to Poornima and Nirmala (2020), a numerical solution is said to be convergent if $\lim_{l \to \infty} \max \left| r\left(x_k\right) - r_k \right| = 0$, where k varies from 1 to N and the error is depicted by the relation $E_{rror} = \left| r\left(x_k\right) - r_k \right|$, where $r\left(x_k\right)$ represent the numerical solution and r_k represent the analytical solution. Based on this fact, the comprehensive analysis of the results obtained from Tables 1, 3, and 5, comparing the numerical solutions using the RK4, BRK5, and "NIM" methods with the analytical solution for problems 1-3, reveals important insights into the performance of these numerical techniques. Additionally, Tables 2, 4, and 6, along with Figures 1-3, provide further clarity by assessing the errors associated with each method.

Numerical Solution Comparison (Tables 1, 3, and 5):

i. The tables show that for all three problems, the RK4, BRK5, and "NIM" methods yield numerical solutions that closely approximate the analytical solution when using a uniform step size of 0.1.
ii. Across the range of X values, the solutions exhibit excellent agreement with the initial conditions and maintain consistency as X increases.
iii. Minor differences between the RK4, BRK5, and "NIM" solutions suggest that these methods perform comparably in approximating the true solution.
iv. The results underscore the effectiveness and accuracy of all three numerical methods in solving the presented problems.

Error Comparison (Tables 2, 4, and 6; Figures 1-3):

i. The tables and graphical representations of errors provide a more in-depth perspective on the performance of each method.
ii. It is evident that the "NIM" method consistently produces more accurate results and exhibits lower error values compared to both the BRK5 and RK4 methods.
iii. Figures 1-3 visually demonstrate that the "NIM" method's error curves approach zero, indicating a convergence to the exact solution as the step size remains constant.
iv. In contrast, the BRK5 and RK4 methods display slightly higher error values, suggesting a marginally less accurate approximation of the true solution.

Findings:

i. The comparative analysis clearly highlights the superiority of the "NIM" method among the three techniques for solving problems 1-3.
ii. Notably, the "NIM" method consistently converges to the analytical solution with minimal error, demonstrating its effectiveness and high accuracy.
iii. While the RK4 and BRK5 methods also provide reliable results, the "NIM" method stands out as the most precise numerical approach for these specific problems.

CONCLUSION

We have solved some first order nonlinear and linear differential equation using the RK4, “NIM” and BRK5 schemes. All the methods provide highly accurate answers for a given first-order nonlinear and linear ordinary differential equation, as proven by the numerical research reported in Section four. Although the "NIM" requires less computational complexity, it is simpler and easier to use. Yet, as can be shown in Tables 1, 3 and 5, the "NIM" is the most effective of the three techniques used in this study. The absolute error plots provide more evidence for this conclusion. As a result, we have been able to demonstrate an approach that is effective in terms of dependability, sufficiency, and simplicity in terms of solving first-order nonlinear and linear ordinary differential problems. "NIM" has been proven efficient and accurate to solve both linear and nonlinear first order ordinary differential equations than RK4 and BRK5 methods. In addition, the study has shown that BRK5 procedure should be utilized to solve first-order ODEs compared to RK4 technique. Further research can be carried out on comparison of numerical methods using "NIM" and other methods apart from RKMs. The results of the assessment provide valuable insights into the strengths and weaknesses of each method. The comparative analysis reveals the relative advantages and disadvantages of the RKMs and the "NIM" in terms of accuracy and computational efficiency. The findings of this research have practical implications for researchers and practitioners in the field of numerical methods for differential equations. They contribute to the understanding of the trade-offs involved in selecting an appropriate numerical method for specific problem types and provide guidance on the application of the new iteration method in practical scenarios.

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